DICTIONARY OF MATHEMATICAL TERMS

DR. ASHOT DJRBASHIAN
in cooperation with
PROF. PETER STATHIS

INTRODUCTION

This dictionary is specifically designed for two-year college students. It contains all the important mathematical terms the students would encounter in any mathematics classes they take. From the most basic mathematical terms of Arithmetic to Algebra, Calculus, Linear Algebra and Differential Equations. In addition we also included most of the terms from Statistics, Business Calculus, Finite Mathematics and some other less commonly taught classes. In a few occasions we went beyond the standard material to satisfy curiosity of some students. Examples are the article "Cantor set" and articles on solutions of cubic and quartic equations.

The organization of the material is strictly alphabetical, not by the topic. There are approximately a total of 1200 entries in this dictionary. Some of them are just simple one-two sentence definitions while others are pretty detailed articles with definitions and even worked examples. All the entries are in lowercase letters unless they contain a proper name. In text, however, some terms might appear in capitalized form in expressions such as Fundamental Theorem of Calculus, Mean value theorem and so on. There are also about eighty illustrations of major functions and geometric figures. Most of the pictures are created by the author and others are imported from open internet sources such as Google and Wikipedia.

Because this is an electronic book and not a traditional hard copy book it is worth writing a few words how we think it should be used the best. The last few years brought lots of changes in how students work with the books and treat them. More and more students opt for electronic books and "carry" them in their laptops, tablets, and even cellphones. This is a trend that seemingly cannot be reversed. This is why we have decided to make this an electronic and not a print book and post it on Mathematics Division site for anybody to download.

Here is how we envision the use of this dictionary. As a student studies at home or in the class he or she may encounter a term which exact meaning is forgotten. Then instead of trying to find explanation in another book (which may or may not be saved) or going to internet sources they just open this dictionary for necessary clarification and explanation.

Why is this a better option in most of the cases? First of all internet search usually results in multiple, sometimes hundreds of choices and that already creates problems. Second, many of these sources are rather general and do not have in mind specifically the level of the students in community colleges. Many of them are actually designed with specialists in mind and contain material that might confuse the student rather than help. As an example, if a student needs to recall the topic "Partial fractions" and ends up in Wikipedia then the explanation would be a very long, general, and confusing one containing lots of unnecessary material. The total number of pages corresponding to that particular entry in Wikipedia (after converting into Word format) is about 12 pages. In contrast, the explanation in this dictionary is very concise, contains only the necessary cases and fits on four Word pages, including worked examples of all cases.

Traditional dictionaries use *italics* to indicate that the given word has its own entry. Electronic books present additional opportunity for the same purpose and especially for quick finding of the indicated terms, namely hyperlinks. In this book we use a combination of both methods and it is worth to clarify further how and when each of these approaches used in our dictionary. As the reader goes through some article in the dictionary he or she might need explanation, clarification, or definition of certain terms used in that article. When we thought that it is important and beneficial to the reader to refresh the memory and clarify that term in order to understand the article at hand we created a hyperlink allowing immediately jump to that term without scrolling through multiple pages. This is similar to most internet sources, such as above mentioned Wikipedia. For example, if the reader sees the expression characteristic equation, then by simply clicking at that word he will jump to page 20 of the text where that term is defined and explained. It is important to note however that not all the terms are hyperlinked and there are multiple reasons for that. First of all, that would make many articles very difficult to read because of abundance of hyperlinks and would destruct reader from the actual article itself. In addition, there are many terms so common that it is obvious that there are specific articles for them. These terms include, but not limited to, words such as Number, Function, Polynomial, Equation, Integral, Derivative, Matrix, etc. In rare cases, however, even some of these terms are hyperlinked by the reason explained above. Because of these in many cases we just used the traditional approach of emphasizing the terms with italics and that also requires some explanation. If the word is in italics then that means one of the following :
1) There is an article for that particular term but in author's view it is not crucial to jump there immediately in order to understand the current article;
2) There may or may not be an article for that term but the definition is contained in this current article the reader is on;
3) There is an article for that term but it is so close (might be the very next or just the previous one) there is no need of hyperlink to get there.
We hope that this hybrid approach would make the use of dictionary as easy as possible.

Dr. Ashot Djrbashian
Mathematics Division
Glendale Community College
Glendale, CA

A

Abel's formula Also called Abel's theorem, for differential equations. Let y_1 and y_2 be solutions of second order equation

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

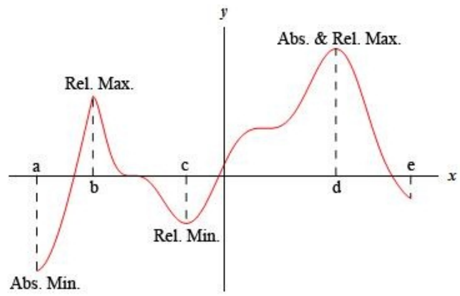
where both p and q are continuous functions on some interval I. Then by Abel's theorem the Wronskian of the equation is given by the formula

$$W(y_1, y_2) = c \exp\left[-\int p(t)\, dt\right],$$

where c is a constant that depends on solutions y_1, y_2 only.

abscissa In the plane Cartesian coordinate system, the name of the x-axis.

absolute maximum and minimum (1) For functions of one variable. A number M in the range of a given function $f(x)$, such that $f(x) \le M$ for all values of x in the domain of f is the absolute maximum. Similarly, a number m in the range of a given function $f(x)$, such that $f(x) \ge m$ for all values of x in the domain of f is the absolute minimum.
If the function f is given on a closed interval $[a, b]$, then to find absolute maximum and minimum values, the following steps to be taken:
(a) Find all critical points (points, where the derivative f' is either zero or does not exist) of the function on the open interval (a, b)
(b) Find the values of f at critical points and endpoints a and b.
(c) Compare all these values to determine the largest and smallest. They will be the absolute maximum and minimum values respectively. See also local maximum and minimum.
(2) For functions of several variables definitions similar to the above are also valid. The critical points are also used to find these maximum and minimum points.

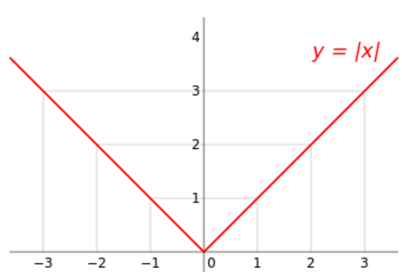


absolute value For a real number r, the non-negative number $|r|$ given by the formula

$$|r| = \begin{cases} r & \text{if} \quad r \ge 0 \\ -r & \text{if} \quad r < 0 \end{cases}$$

Absolute value of a real number indicates the distance between the point on the *number line* corresponding to that number and the origin.

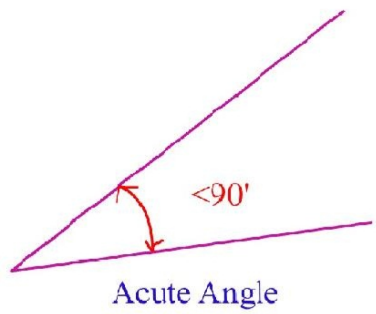
absolute value function The function $f(x) = |x|$, defined for all real values x.



absolutely convergent series A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series formed with the absolute values of its terms $\sum_{n=1}^{\infty} |a_n|$ converges. If a series is absolutely convergent, then any rearrangement of that series is also convergent and the sum is the same as the original series.

acceleration In physics, the rate of change of the velocity of a moving object. Mathematically, if the distance traveled by the object is given by the function $s(t)$, then acceleration is the second derivative of this function: $a(t) = s''(t)$.

acute angle An angle that measures between 0 and 90° in degree measure or between 0 and $\pi/2$ in radian measure.



addends In arithmetic, two or more numbers to be added. Similarly, in algebra, two or more expressions to be added.

addition One of the four basic operations in arithmetic and algebra. By adding two (or more) numbers we determine the sum of those numbers.

addition and subtraction of complex numbers If $a + ib$ and $c + id$ are two complex numbers, then their sum or difference is defined to be another complex number, where the real part is the sum or difference of the real parts of the given numbers and the imaginary part is the sum or difference of imaginary parts: $(a + ib) \pm (c + id) = (a \pm c) + i(b \pm d)$.

addition and subtraction of fractions If $\frac{a}{b}$ and $\frac{c}{d}$ are two numeric fractions, then their sum or difference is formally defined to be the fraction

$$\frac{ad \pm cb}{bd}.$$

In practice, to make calculations simpler, we first find the least common denominator (LCD) of two denominators of given fractions, substitute both fractions by equivalent fractions with the found LCD, and then add or subtract these two fractions. Example: to add $\frac{3}{8} + \frac{5}{6}$, we find the LCD of 8 and 6, which is 24, substitute the given fractions by $\frac{9}{24} + \frac{20}{24} = \frac{29}{24}$.

addition and subtraction of functions Let $f(x)$, $g(x)$ be two functions. Then the "sum" function $(f + g)(x)$ is defined to be the sum of their values: $(f + g)(x) = f(x) + g(x)$. Similarly, $(f - g)(x) = f(x) - g(x)$.

addition and subtraction of matrices The sum of two matrices is defined only if they have the same dimensions: the same number of rows and columns. If A and B are two $m \times n$-matrices, then their sum is defined to be the matrix C which has as elements the sums of elements of the given matrices in each position. Example:

$$\begin{pmatrix} 1 & 1 & -2 \\ -1 & 0 & 4 \\ 5 & -2 & 0 \end{pmatrix} + \begin{pmatrix} -2 & 2 & 3 \\ 1 & -3 & -4 \\ -4 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 3 & 1 \\ 0 & -3 & 0 \\ 1 & -2 & 1 \end{pmatrix}.$$

Subtraction of matrices is defined similarly.

addition and subtraction formulas In trigonometry, the formulas

$$\cos(\phi + \theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$

$$\cos(\phi - \theta) = \cos\phi\cos\theta + \sin\phi\sin\theta$$

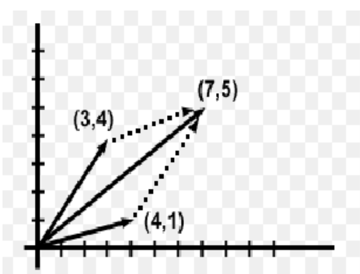
$$\sin(\phi + \theta) = \cos\phi\sin\theta + \sin\phi\cos\theta$$

$$\sin(\phi - \theta) = \cos\phi\sin\theta - \sin\phi\cos\theta$$

$$\tan(\phi + \theta) = \frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta}$$

$$\tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}$$

Similar formulas also exist for other trigonometric functions but are very rarely used.



addition and subtraction of vectors Let $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ be two vectors in some n-dimensional vector space V. To add or subtract these two vectors, we just add or subtract their corresponding coordinates:

$$\mathbf{u} \pm v = (u_1 \pm v_1, u_2 \pm v_2, \cdots, u_n \pm v_n).$$

In case of vectors in the plane, addition of vectors has very simple geometric interpretation. It could be described by the triangle law (also called *parallelogram law*): Place the initial point of the second vector on the terminal point of the first vector. The vector connecting the initial point of the first one with the terminal point of the second will be the sum of these vectors.

addition property of equality Let A, B, C be any algebraic expressions. If $A = B$, then $A + C = B + C$.

addition property of inequalities Let A, B, C be any algebraic expressions. If $A \leq B$, then $A+C \leq B + C$.

addition rule for probabilities (1) If two events, A and B, are disjoint, then the addition rule is

$$P(A \text{ or } B) = P(A) + P(B).$$

(2) In the more general case when two events might have a common outcome, the addition rule is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

Note also that the set $\{A \text{ or } B\}$ is the union of the sets A and B and is denoted by $A \cup B$ and the set $\{A \text{ and } B\}$ is the intersection of that sets and is denoted by $A \cap B$.

additive identity An element e of some set S,that has the property $a + e = a$ for any element a. In the number system (real or complex), $e = 0$.

additive inverse For a given element a of some set S, the element b with the property $a + b = e$, where e is the additive identity. In the number system (real or complex), $e = 0$ and $b = -a$.

adjacent angle Two angles that have common vertex, one common side but no common interior points.

adjacent side Any of two sides in a triangle which make up the angle, are called adjacent. The other side of the same triangle is the opposite side. In the trigonometry of right triangles, this term is reserved for one side of acute angle only, namely the side which is the leg, not the hypotenuse of the triangle.

adjoint matrix Let $A = (a_{ij})$ be a *square matrix* with real or complex entries and denote by C_{ij} the cofactor of the element a_{ij}. Then the matrix made up by all cofactors,

$$\begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}$$

is called the adjoint of the given matrix and denoted by $\text{adj}(A)$. The adjoint matrix allows computation of the inverse of the matrix A by the formula

$$A^{-1} = \frac{1}{\det(A)} adj(A).$$

For a more practical way of determining the inverse matrix see the corresponding entry.

Airy's equation The second order equation

$$y'' - xy = 0, \quad -\infty < x < \infty.$$

See series solution for exact form of the solution of this equation.

algebra One of the main branches of mathematics, that deals with numbers, variables, equations, etc. and operations with them. In a more general setting, the developments of classical algebra, namely linear algebra and abstract algebra, deal with vectors, matrices and structures such as groups, rings, fields and others.

algebra of matrices The term is understood as the collection of rules for operations with matrices: addition, scalar multiplication, multiplication of matrices, and inverses of matrices.

algebraic equation An equation involving just *algebraic function*. Most common example is an algebraic equation involving a polynomial: $P(x) = 0$ or $P(x_1, ..., x_n) = 0$, where P is a polynomial of one or n variables respectively. Example: $4x^5 - 3x^4 + x^2 - 3x + 5 = 0$.
Additionally, algebraic equations include also *rational* and *radical* equations.

algebraic expression An expression which involves numeric constants, one or more variables and only the algebraic operations of addition, subtraction, multiplication, division, and root extraction.

algebraic function A function that involves the algebraic operations of addition, subtraction, multiplication, division, and root extraction. The function

$$f(x) = \frac{\sqrt[3]{3x^2 + 2x - 5}}{x^4 - 2x^3 + 1}$$

is an algebraic function but $g(x) = \log(\sin x)$ is not.

algebraic number A complex number z is an algebraic number if it satisfies a non-trivial polynomial equation $P(z) = 0$ for which the coefficients are rational numbers. It is known that many irrational numbers are not algebraic, in particular numbers π and e. These kind of numbers are called transcendental

algebraic operation One of the basic operations of algebra: addition, subtraction, multiplication, division, or root extraction.

algorithm A method or procedure of solving a given problem or whole class of problems by a limited number of standard operations.

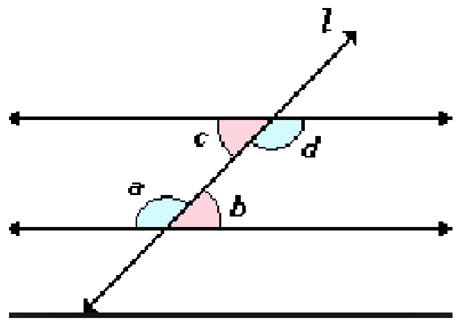
almost linear system In differential equations. Let A be a *square matrix* and $\mathbf{x}$ be a *vector-function.* The system of equations

$$\mathbf{x}' = A\mathbf{x} + \mathbf{g}(x)$$

is called almost linear, if the vector-function $\mathbf{g}$ is small compared to $\mathbf{x}$, i.e., if

$$\frac{||\mathbf{g}(x)||}{||\mathbf{x}||} \to 0, \qquad \mathbf{x} \to 0.$$

alternate angles When two parallel lines are crossed by the third one (called a transversal), eight angles are formed. Two pairs of non-adjacent interior angles are called alternate. Alternate angles are always equal. In picture below pairs a and d and b and c are alternating angles.



alternating harmonic series The *convergent series* $\sum_{n=1}^{\infty}(-1)^{n+1}/n$.

alternating series A numeric series where the successive terms have opposite signs.

alternating series estimation theorem Let $S = \sum(-1)^{n-1}b_n$ be the sum of an alternating series satisfying conditions

$$0 \le b_{n+1} \le b_n, \quad \lim_{n\to 0} b_n = 0.$$

Then the *remainder* R_n of the series satisfies the inequality $|R_n| = |S - S_n| \le b_{n+1}$. See also error estimate for alternating series.

alternating series test If $\sum_{n=1}^{\infty} a_n$ is an alternating series with

$$(i)|a_{n+1}| < |a_n| \quad \text{for all n and}$$

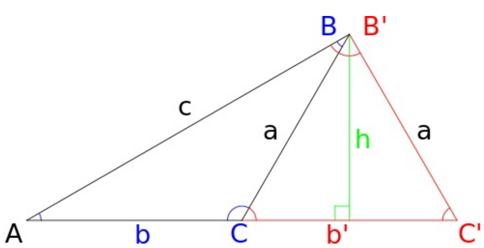
$$(ii) \lim_{n\to\infty} |a_n| = 0,$$

then the series is convergent.

alternative hypothesis A hypothesis (statement) that is contrary to the original (null) hypothesis. Notations for this hypothesis are H_a or H_1. The form of the alternative hypothesis varies depending on the form of the null hypothesis. If the null hypothesis has the form $H_0 : \mu = \mu_0$, then for the alternative hypothesis we have three possible forms: $H_a : \mu \ne \mu_0, \mu < \mu_0$, or $\mu > \mu_0$. Accordingly, we have two-sided (or two-tailed), left-sided, or right-sided alternative hypothesis. Some authors also accept different forms of null hypothesis, specifically $H_0 : \mu \leq \mu_0$ or $\mu \geq \mu_0$. In this situation the alternative hypothesis will change accordingly. See also null hypothesis, hypothesis testing.

altitude of a triangle The perpendicular segment from any vertex of a triangle to the opposite side (or its continuation). Depending on the type of triangle (acute, right, obtuse), the altitude may or may not be inside the triangle.

ambiguous case One of the cases of solving triangles when two sides of a triangle and one angle not formed by these sides are given (the case SSA). In this case three outcomes are possible: triangle has no solution, triangle has one definite solution, and triangle has two definite solutions. In the following examples the angles are denoted by A, B, C and the sides are denoted by a, b, c and we follow the convention that side a is opposite to angle A, side b is opposite to angle B and side c is opposite to angle C.



(1) Assume we know that $a = 15,\ b = 25, A = 85°$. Using the Law of sines we get

$$\sin B = b\frac{\sin A}{a} \approx 1.66 > 1,$$

which is impossible. This means that it is not possible to construct a triangle with the given values.
(2) Let $a = 22,\ b = 12,\ A = 42°$. Here the use of the Law of sines gives $\sin B \approx 0.365$ and solving this equation we get two solutions: $B \approx 21.4°$ and $B \approx 158.6°$. The second angle is not possible because otherwise the sum of angles would have been greater than 180°, hence the first one is valid only. From here, $C \approx 116.6°$. The remaining element of the triangle, the side c is found by another use of the Law of sines: $c = a \sin C/ \sin A \approx 29.4$ and the triangle has unique solution.
(3) Here $a = 12,\ b = 31, ; A = 20.5°$. As in the previous case we find two solutions for the angle $B:\ B_1 \approx 64.8°,\ B_2 \approx 115.2°$. Unlike the previous case, here both are valid solutions. Continuing again as in the previous case we get also two solutions for the remaining elements. This means that there are two possible triangles with the given data (information). For the first one we have $B_1 \approx 64.8°,\ C_1 \approx 94.7°,\ c_1 \approx 34.15$, and for the second one $B_2 \approx 115.2°,\ C_2 \approx 44.3°,\ c_2 \approx 23.93$. See also solving triangles for other possible cases.

amplitude of a graph For a graph of some *periodic function* the half of the difference between absolute maximum and absolute minimum values. This notion is mostly used for trigonometric functions. Example: For the graph of the function $f(x) = 3 \sin x$ the highest point (maximum) is 3 and the lowest point (minimum) is -3. So, the amplitude is $a = (3 - (-3))/2 = 3$.

amplitude of trigonometric function For the functions sin and cos, the half of the difference between absolute maximum and absolute minimum values. Hence, for the function $a \sin(bt + c) + d$, the amplitude will be $|a|$.

analytic function For real-valued functions of one variable: the function is called analytic at some point $x = a$, if it is infinitely differentiable on some neighborhood of that point and its Taylor-MacLaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

converges in a neighborhood of that point to the function itself. Not all infinitely differentiable functions are analytic. The function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

is infinitely differentiable, but all of its Taylor-MacLaurin coefficients at the origin are zero and hence the series does not represent the function f.

analytic geometry One of the branches of geometry. Analytic geometry combines the geometric and algebraic methods by introducing coordinate systems, and presenting geometric objects with the help of algebraic functions and their *graphs*.

angle Geometric object, which consists of a point, called *vertex* and two *rays* coming out of this point. In trigonometry and calculus, angles usually are placed in *standard position*, when the vertex is placed at the origin of the Cartesian coordinate system, one side, called *initial*, coincides with the positive half of the x-axis and the other side, called *terminal*, can have arbitrary direction. In this position angles can be both positive and negative. The angles, where the terminal side is moved counterclockwise, are considered positive and clockwise direction is considered negative.
The two main measuring units for angles are the degree measure and the radian measure. Two angles with the same terminal side are called *coterminal* and they differ in size by an integer multiple of a full circle, i.e. by multiple of $360°$ or 2π, depending on measuring unit. Angles traditionally are classified by their measure: An angle between 0 and 90 degrees is called *acute*, a 90 degree angle is called *right*, between 90 and 180 degrees – *obtuse* and 180 degrees– *straight*. Additionally, angles measuring 90, 180, 270 or 360 degrees are called *quadrantal*.

angle between curves is defined to be the angle between tangent lines to these curves at the point of intersection.

angle between vectors For vectors in *inner product spaces*. If $\mathbf{u}, \mathbf{v}$ are two vectors in the inner product vector space V, then the angle θ between them is determined from the relation

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{||\mathbf{u}||||\mathbf{v}||},$$

where $\langle \mathbf{u}, \mathbf{v} \rangle$ is the inner product in V and $||\mathbf{u}||$ denotes the norm (magnitude, length) of the vector.

angle between vector and plane is defined to be the *complement* of the angle between that vector and the *normal vector* to the plane.

angle bisector See bisector of an angle.

angle of depression If an object is viewed from above, then the angle between the horizontal line and the line drown between the viewer and the object, is the angle of depression.

angle of elevation If an object is viewed from below, then the angle between the horizontal line and the line drown between the viewer and the object, is the angle of elevation.

ANOVA Stands for Analysis of Variance. It is a collection of statistical methods designed to compare *means* of two or more groups of values and generalizes the normal distribution tests and t-tests.

angular speed The rate of change of the angle as an object moves along a circle. If the angle is measured in *radian measure* and is denoted by θ, the *radius* of the circle is r, then the angular speed is given by the formula $\omega = \frac{\theta}{r}$.

annihilator A differential operator is an annihilator for some function, if application of that operator to the function results in zero function. Example: The operator $D^2 + 4$ is an annihilator for the function $\sin 2x$, because

$$(D^2 + 4)\sin 2x = \frac{d^2}{dx^2}\sin 2x + 4\sin 2x = 0.$$

annual percentage rate The rate banks charge for borrowed money, or pay for the money deposited. Expressed in percentages or equivalent decimal form. For example, annual percentage rate of 9 percent is translated into the numeric value 0.09, and APR 5.7 percent becomes 0.057.

annulus A region bounded by two concentric circles of different radiuses. Also called ring or washer.

antiderivative Another name for *indefinite integral*.The antiderivative of a function is a new function which has that given function as its derivative.

antidifferentiation Same as finding the *indefinite integral*.

antilogarithm For a number y and a base b, the number x such that $\log_b x = y$. Same as exponential function.

approximate (1)The procedure of substituting for the exact value of a number with a value close to the exact value; (2) The procedure of substituting for the values of a function by values of other, usually simpler, functions. Most often the approximations are done by polynomials or rational functions because of the simple nature of these functions.

approximate integration The procedure of calculating *definite integrals*, when exact integration is impossible or difficult. There are various methods of approximate integration. For specific methods see midpoint rule, Simpson's rule, trapezoidal rule, and approximate integration by Riemann sums below.

approximate integration by Riemann sums By the definition of definite integral, it is the limit of Riemann sums, as the length of interval partitions approaches zero. Hence, each Riemann sum represents an approximate value of the given integral. Formally, let the function $f(x)$ be defined on some interval $[a, b]$ and form the left Riemann sum

$$R_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

Then the number R_n is the left endpoint approximation of the integral $\int_a^b f(x)dx$. The right endpoint approximation is defined similarly.

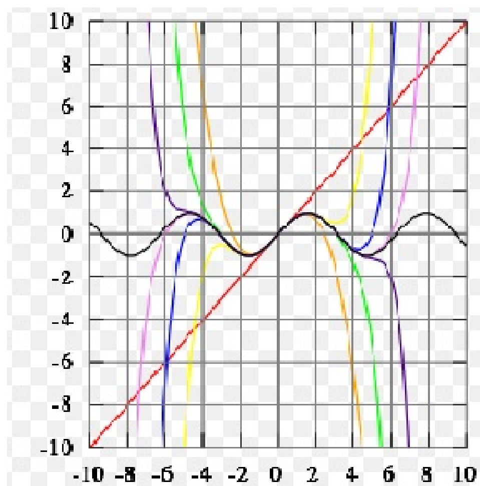
approximation Collective term for different methods allowing to *approximate* numbers, functions, and solutions of different types of equations, when exact numbers or solutions are not available. For more details see linear approximation, quadratic approximation, Newton's method, and approximation by Taylor polynomials below.

approximation by differentials For a given differentiable function $f(x)$ around some point $x = a$, the simplest approximation is by a line passing through that point and with *slope* equal to the derivative of the function at that point: $f'(a)$. The equation of that line is given by

$$L(x) = f(a) + f'(a)(x - a).$$

For values not too far from a this approximation gives satisfactory results. Also called linear approximation.

approximation by Taylor polynomials If a function $f(x)$ can be expanded into a Taylor series



that converges to that function, then the *partial sums* of that series become approximations to the function. Let

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(j)}(a)}{j!}(x-a)^j$$

be that expansion around the point $x = a$. If f is bounded by $M > 0$ in the interval $|x - a| \le d$ and

$$R_n(x) = \sum_{j=n+1}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is the remainder of the series, then

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!}, \quad |x - a| \le d.$$

This inequality, called Taylor's inequality, tells that the first n terms of the series represent a good approximation for the function f.
The graph in this article shows Taylor approximations of the function *sinx* by Taylor polynomials of degree 1,2,3,4 and 5.

approximation problems Different problems that come down to substituting the exact value of some quantity by a value different but close to the actual value.

arc Usually, a connected portion of a circle. In calculus, a connected portion of any smooth curve.

arc length formula (1) For the circle. If the radius is r and an arc is subtended over the angle of measure θ (in radian measure), then the length of the arc is given by the formula $s = r\theta$.
(2) For a general curve on the plane given by the function $y = f(x)$ on interval $[a, b]$. The arc length is given by the formula

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

(3) For a curve in R^3 given by *parametric equations* $x = x(t),\ y = y(t),\ z = z(t)$, where $a \leq t \leq b$, the arc length is given by a similar formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

Formula for the parametric plane curve is similar, with the last term missing.

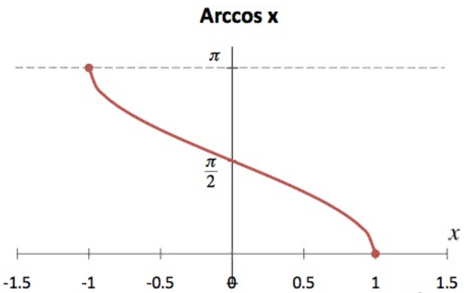
arc length function The expression

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt,$$

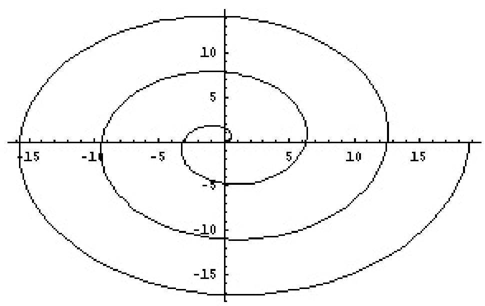
which comes out in calculations of arc lengths, is the arc length function.

arccosine function The inverse of the $\cos x$ function, denoted $\cos^{-1} x$ or $\arccos x$. Represents the angle θ on the interval $[0, \pi]$ such that $\cos\theta = x$. The domain of this function is $[-1, 1]$ and the range is $[0, \pi]$. We have the following differentiation formula:

$$\frac{d}{dx}\cos^{-1} = -\frac{1}{\sqrt{1 - x^2}}.$$

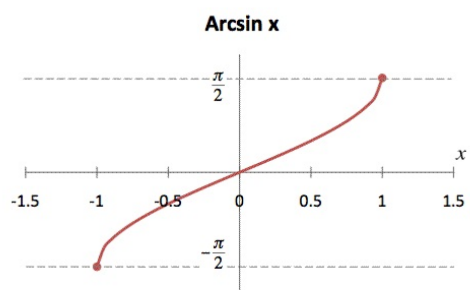


Archimedian spiral The polar curve $r = \theta e^{i\theta}$. The parametric equation could be written as $x(t) = t\cos t, y(t) = t\sin t$.



arcsine function The inverse of the $\sin x$ function, denoted either $\sin^{-1} x$ or $\arcsin x$. Represents the angle θ on the interval $[-\pi/2, \pi/2]$ such that $\sin\theta = x$. The *domain* of this function is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$. We have the following differentiation formula:

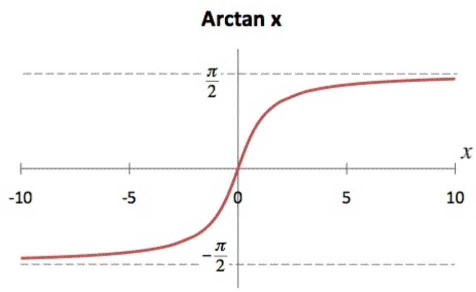
$$\frac{d}{dx}\sin^{-1} = \frac{1}{\sqrt{1 - x^2}}.$$



arctangent function The inverse of the $\tan x$ function, denoted $\tan^{-1} x$ or $\arctan x$. Represents the angle θ on the interval $(-\pi/2, \pi/2)$ such that $\tan\theta = x$. The *domain* of this function is $(-\infty, \infty)$ and the range is $(-\pi/2, \pi/2)$. The inverse functions

$\cot^{-1} x$, $\sec^{-1} x$, $\csc^{-1} x$ are defined in a similar manner. We have the following differentiation formula:

$$\frac{d}{dx}\tan^{-1} = \frac{1}{1+x^2}.$$



area One of the basic notions of geometry used in many branches of mathematics. Indicates the "amount of plane" included in a certain plane region.

area between curves Let two curves be given by the functions $f(x)$ and $g(x)$. Then the area between that curves (on some interval $[a, b]$) is given by

$$\int_a^b |f(x) - g(x)|dx.$$

area function For a given non-negative function $f(x)$ defined and continuous on some interval $[a, b]$, the integral

$$A(x) = \int_a^x f(t)\, dt$$

is sometimes called area function, because it represents the area under the curve $y = f(x)$ from point a to x.

area of a circle For a circle with radius r, the area is given by the formula $A = \pi r^2$.

area of an ellipse The area of an ellipse is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is equal to πab. Because the area does not change during translation, the same value for the area is also true for translated ellipse.

area of geometric figures For a rectangle with width w and length ℓ, $A = w \cdot \ell$. In the particular case when the length and width are equal, we get a square (of side a, for example) and the area is $A = a^2$.
For a triangle with base b and height h, $A = \frac{1}{2}b \cdot h$
For a parallelogram with base a and height h, $A = b \cdot h$
For a trapezoid with two parallel bases a and b and height h, $A = \frac{a+b}{2} \cdot h$.
In general, the area of any *polygon* could be calculated by dividing it into a number of triangles or rectangles and adding the areas of these smaller pieces. See also the corresponding definitions for pictures and notations.

area in polar coordinates If a *curve* is given with the polar equation $r = f(\theta)$, where $a \le \theta \le b$, then the area under that curve is equal to

$$A = \frac{1}{2}\int_a^b [f(\theta)]^2 d\theta.$$

area of a sector The area of a *sector* of a circle with radius r and the opening angle (in radian measure) θ is given by the formula $A = \frac{1}{2}r^2\theta$.

area of a surface See surface area.

area of a surface of revolution Assume we have a surface obtained by revolving the smooth curve $y = f(x)$, $a \le x \le b$, about some line. Then, denoting by r the distance between arbitrary point x on interval $[a, b]$ and the line, the surface area can be expressed as

$$S = \int_a^b 2\pi r\sqrt{1 + (f'(x))^2}dx.$$

area under the curve If a curve is given by the function $f(x) \ge 0$, then the area under that curve from point a to point b is given by the definite integral

$$\int_a^b f(x)dx.$$

area under a parametric curve Suppose the *curve* $y = F(x)$, $F(x) \geq 0$, is written in parametric form $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$. Then the area under that curve (under the condition that this curve traverses only once as t increases from α to β) is given by the formula

$$A = \int_\alpha^\beta g(t)f'(t)dt.$$

argument of a complex number In the trigonometric representation of a complex number $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ the angle θ.

argument of a function In a function $y = f(x)$ the independent variable x. Also, in a function of any number of variables, $y = f(x_1, x_2, \cdots, x_n)$, the independent variables $x_1, x_2, \cdots, x_n$.

arithmetic The earliest and most fundamental branch of mathematics. Arithmetic deals with *numbers* and arithmetic operations on them: addition, subtraction, multiplication and division. In a more modern and wider sense, the term arithmetic is also used as a synonym to number theory, which is the only part of arithmetic still subject to research activity.

arithmetic average or mean For a positive integer n, and given real numbers $a_1, a_2, \cdots, a_n$, the mean is $(a_1 + \cdots + a_n)/n$. Example: The average (mean) of the set of numbers $\{1, 3, -2, 5, 8, -4, 6, -5, 12\}$ is

$$m = \frac{1 + 3 - 2 + 5 + 8 - 4 + 6 - 5 + 12}{8} = 3.$$

arithmetic expression An expression which involves numeric constants and the arithmetic operations of addition, subtraction, multiplication, division, and natural exponents. Example:

$$\frac{7^2 - [(9-7)^2 + 5 \cdot 2]}{6^2 - 2^2}.$$

arithmetic-geometric mean inequality The inequality $\sqrt{a \cdot b} \leq \frac{a+b}{2}$, which is true for any non-negative numbers a and b. In the most general case the inequality is as follows: Let $a_1, a_2, \cdots, a_n$ be any non-negative numbers. Then

$$\sqrt[n]{a_1 \cdot a_2 \cdots a_n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

arithmetic growth A quantity that grows according to the arithmetic progression (see below) rule: $a_n = a_0 + (n-1)d$ with some real number d.

arithmetic sequence (progression) A sequence of real numbers where the difference between two consecutive terms is the same. The general term of such sequence is given by the formula $a_n = a_1 + (n-1)d$, where d is the difference between two consecutive terms. The *sum* of the first n terms of arithmetic sequence is given by the formula

$$S_n = \sum_{i=1}^n a_i = \frac{a_1 + a_n}{2} n.$$

associated quadratic form For any *quadratic form*, the associated quadratic form is the quadratic form with the same second degree terms and no first degree terms or constant. For quadratic form in two variables $ax^2 + 2bxy + cy^2 + dx + ey + f$, the associated quadratic form would be $ax^2 + 2bxy + cy^2$. The case of more variables is defined similarly.

association Any kind of relationship between two sets of variables or values. For example, in the function $y = 2x + 1$ the variables x and y are related by the given formula and the values of y are associated with the values of x so that to each x the double of that value plus one more is associated. Similarly, if two sets of values forming *ordered pairs* are put into a scatterplot, then by looking at the general direction we might see a positive association (bigger y's correspond to bigger x's) or a negative one (bigger y's correspond to smaller x's).

associative property (1) For addition: for any numbers a, b, c,complex or real, $a+(b+c) = (a+b)+c$. (2) For multiplication: for the same three numbers, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

astroid Let a circle of radius r roll inside another circle of radius $4r$. A point on a smaller circle will leave a trace that is called astroid. Parametric equations are given by

$$x = \cos^3\theta, \quad y = \sin^3\theta.$$

Astroid is a special case of the hypocycloid.

asymptote As a rule, a straight line with the property that the graph of a given function approaches it. There are three types of line asymptotes. Vertical asymptotes have the form $x = a$, a is real, horizontal asymptotes are of the form $y = b, b$ is real and slant asymptotes are of the form $y = ax + b$ with $a \neq 0, \infty$. Vertical asymptotes are characterized by the fact that $\lim_{x\to a^\pm} f(x) = \pm\infty$ and the function f can never cross that line. For horizontal asymptotes we have $\lim_{x\to\pm\infty} f(x) = b$. Here the function may or may not cross the asymptote line. For the slant asymptotes we have $\lim_{x\to\infty}[f(x) - (ax + b)] = 0$ or the similar relationship with $x \to -\infty$. Here also, the function may or may not cross the line at some (possibly infinitely many) points. Slant asymptotes are also called *oblique*.
In addition to line asymptotes also non-linear asymptotes are sometimes considered, such as quadratic or cubic.

asymptote of a hyperbola See hyperbola.

asymptotic curve is a curve always tangent to an asymptotic direction of the surface. An asymptotic direction is one in which the normal curvature is zero.

augmented matrix For the system of m equations with n unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

the $m \times (n + 1)$-matrix of all coefficients with the addition of constant terms column:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}.$$

Augmented matrices are used to solve systems of linear equations with Gaussian or Gauss-Jordan elimination methods.

autonomous equation First order differential equation of the form

$$\frac{d}{dt}x(t) = f(x(t)).$$

These kind of equations describe particles which rate of motion depends only on their position but not of time.

autonomous system System of autonomous equations.

auxiliary equation For differential equations, the corresponding algebraic equation. For example, the equation $ay'' + by' + cy = 0$ will have the auxiliary equation $ar^2 + br + c = 0$. Also is called characteristic equation .

average cost function Let x denote the number of units of any product by some company. Then $C(x)$ denotes the cost function. The average cost function is defined to be the quantity $c(x) = C(x)/x$. This function is very important in economics.

average rate of change If a function $f(x)$ is defined on some interval $[a, b]$, then the average rate of change of that function on any sub-interval $[c, d]$ of the given interval is the quantity $(f(c)-f(d))/(d-c)$. This quantity is used to find the instantaneous rate of change by making the interval $[c, d]$ smaller and eventually approaching its length to zero.

average value of a function over an interval $[a, b]$ is defined by the formula

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx.$$

average velocity For a moving object the average velocity in the time interval between t_1 and t_2 is determined by the formula $v = (s_2 - s_1)/(t_2 - t_1)$, where s_1 and s_2 are the distances traveled by the object at the moments t_1 and t_2 correspondingly.

axiom A statement that is assumed to be true without proof, and used as a basis for proving other statements, less obvious. Also is called *postulate*.

axis (1) In Cartesian coordinate system, the name of lines that are used to calculate the position of a point in two, three or higher dimensional spaces. See also *x-axis, y-axis, z-axis*. The x-axis is also called abscissa and real axis and the y-axis is called ordinate and imaginary axis.
(2) If a curve or a surface is symmetric with respect to some line, then this line is called axis of symmetry or rotation. Example: The y-axis is the axis of symmetry for the parabola $y = x^2$.
(3) In polar coordinate system the ray coming out from the pole is called polar axis.

B

back-substitution The last stage of the solution of systems of linear equations by Gaussian elimination method. After applying that method to a $n \times n$ system, we arrive to triangular system

$$x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$

$$x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

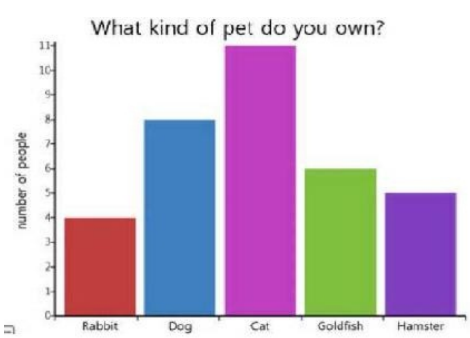
$$x_{n-1} + a_{n-1n}x_n = b_{n-1}$$

$$x_n = b_n.$$

Now, the back substitution consists of the following steps. First, we have $x_n = b_n$. Substituting back into the previous equation, we get $x_{n-1} = b_{n-1} - a_{n-1n}b_n$. Continuing this way up, we eventually find all the values of variables $x_1, x_2, \cdots, x_{n-1}, x_n$, thus completing solution of the system.

backward phase The second stage (or phase) of the solution of linear systems by Gaussian elimination. See back substitution above for details.

bar graphs Also called bar chart. Bar graphs are



one of many ways of visual representation of qualitative (categorical) data. In order to distinguish them from histograms, a little space is left between bars in the graph. The picture above shows the production numbers of some hypothetical company categorized by months.

base of exponential function In exponential function $y = a^x$, where $0 < a < \infty$, $a \neq 1$, and $-\infty < x < \infty$, the constant a is called the base and the variable x is called the *exponent*

base of logarithm In logarithmic function $y = \log_b x$, where $0 < b < \infty$, $b \neq 1$, and $0 < x < \infty$, the constant b is called the base of logarithm.

basis Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ be a subset of the linear vector space V. S is called a basis for V, if for any given vector $\mathbf{u}$ in V, there exists a unique collection of scalars $c_1, c_2, \cdots, c_n$, such that $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$. These numbers are called coordinates (or components) relative to the given basis. A vector space has infinitely many bases and changing the basis changes also the coordinates of any given vector. Any basis is necessarily linearly independent. The number of the vectors in the set S is called *dimension* of the space V. Similar definition works also in cases when the dimension is not finite. A basis is called orthogonal, if the inner products of all possible pairs (of different vectors) in that basis are zeros:

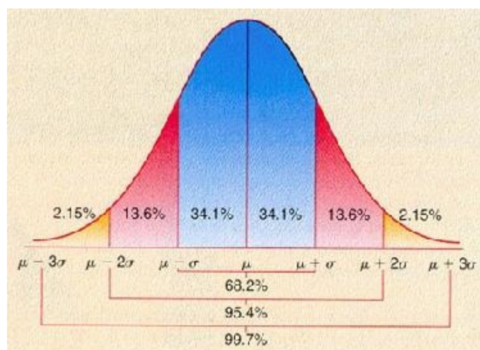
$$\mathbf{x}_i \cdot \mathbf{x}_j = 0, \quad 1 \leq i, j \leq n, \quad i \neq j.$$

If, additionally, $\mathbf{x}_i \cdot \mathbf{x}_i = 1$, $1 \leq i \leq n$, then the basis is called orthonormal. Example: The system of vectors $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ is a basis in the three dimensional Euclidean space R^3 which is also orthonormal.
The basis could be defined and found not only for the whole space, but also for any subspace of the given space V. For example, the nullspace of a matrix or transformation also has a basis because it is a subspace of the given space. Also, basis for a *complex vector space* is defined similarly with exception of a necessary change in the part of orthogonality.

bearings Method of measuring angles used primarily in navigation. There are at least two approaches. First, the angles are measured with respect to vertical axis. In this situation all angles could be made acute if we measure starting from the ray that comes out from the origin and either goes up (North direction) or down (South direction). So, the notation $N52°E$ means the angle is 52 degrees measured from positive y-axis clockwise (to the East) and notation $S47°W$ means that the angle is 47 degrees measured from negative y-axis clockwise (to the West). This method is used traditionally in marine navigation. The second method is mostly used in areal navigation and has only one starting direction (initial side of the angle), namely, the positive y-axis and the angles measured vary between 0 and 180 degrees.

bell-shaped (Gaussian) curve The curve describing the normal distribution. Mathematically the function having that graph is given by the formula

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$



where μ and σ are the mean and standard deviation of the distribution respectively. The graph of this distribution looks like the one above.

Bernoulli equations The first order non-linear equations

$$y' + P(x)y = Q(x)y^n.$$

These equations can be transformed into linear equations

$$\frac{w'}{1-n} + P(x)w = Q(x)$$

with the substitution $w = y^{1-n}$, $n \neq 0, 1$, and solved using integrating factors.

Bessel equation In differential equations. The series of second order linear homogeneous equations

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

with parameter ν. Depending on the value of this parameter we get Bessel equations of order zero, one-half, one, two, and, in general, ν. The solutions are found by the method of series solution and heavily depend on the value of the parameter. See Bessel functions below.

Bessel functions These functions arise during the series solution of the *Bessel equations*. Depending on the parameter ν, there could be many different Bessel functions. Additionally, this functions are categorized into first kind and second kind. The functions

$$J_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

and

$$J_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n+1)!n!2^{2n}}$$

are called Bessel functions of the first kind of order zero and one respectively and they are the solutions of Bessel equations of order zero and one. The Bessel functions of second kind differ from functions of the first kind primarily by the presence of a logarithmic term. For example, the Bessel function of the second kind and order zero $Y_0(x)$ is given by the representation

$$\frac{2}{\pi} \left[\left(\gamma + \ln \frac{x}{2} \right) J_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} H_m}{2^{2m}(m!)^2} x^{2m} \right]$$

where γ is the Euler's constant and H_m is the mth partial sum of harmonic series. Now, the general solution of the Bessel equation of order zero is given by the function

$$y = c_1 J_0(x) + c_2 Y_0(x)$$

with arbitrary constants $c_1,\ c_2$.

best approximation An approximation that provides the closest possible value to a quantity we wish to approximate. Depending on nature of the object, best approximation will be found by a specific method.

best approximation theorem Let W be a finite-dimensional subspace of an inner product space V and let $\mathbf{u}$ be some vector in V. Then the vector $proj_W \mathbf{u}$ is the best approximation from W to $\mathbf{u}$ in the sense that

$$||\mathbf{u} - proj_W \mathbf{u}|| \leq ||\mathbf{u} - \mathbf{w}||$$

for any vector $\mathbf{w}$ that is in W.

best fitting line A line that best describes the *paired data* collected from sample and presented by a scatterplot. See least squares regression line.

bias In sampling or statistical experiment. If by some reason some statistic of collected data from sampling or experiment has the tendency of systematically differing from population *parameter*, then it is called biased. The reason of bias could be subjective (such as bad methodology of data collection) or objective (some statistics are always biased).

biconditional statement A logical statement of the form "If A then B and if B,then A",or, equivalently, "A if and only if B". This statement also can be written symbolically as $A \Leftrightarrow B$. Examples can be: "A triangle is equilateral if and only if it has three equal angles", or "A number is divisible by two if and only if it is even". See also conditional statement.

binary numbers The binary numeral system represents numeric values using only two symbols, usually 0 and 1. In this system the sequence of whole numbers 0,1,2,3,... is represented by the following sequence: 0,1,10,11,100,101,110,111,1000,... The use of this system is crucial in computers and other electronic devices because it allows to represent all real numbers (or their approximations) with the help of signal-no signal sequences. See also decimal numbers.

binary operation An operation involving two mathematical objects. Examples of binary operations are addition and multiplication of numbers, functions, matrices, etc.

binomial A polynomial of one or more variables that has exactly two terms, called *monomials*. Examples are $2x + 1,\ 3x^3 + 4x^2,\ 2x^4y^3 + 5xy^2z$.

binomial coefficients In *binomial expansion* (see below), the numeric coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $n \geq 1$ is an integer and $0 \leq k \leq n$ is another integer.

binomial distribution Suppose we are conducting some experiment and the following conditions are satisfies:
1) There are n repeated trials every time;
2) Each trial has exactly two possible outcomes, called success (S) and failure (F);
3) Probability of S is p and probability of F is $q = 1 - p$;
4) The random variable x is the number of successes in n trials and takes values from 0 to n.
In this case it is said that we have binomial setting. Binomial distribution is the distribution of probabilities in this setting and is given by the formula

$$P(x = k) = \binom{n}{k} p^k q^{n-k},$$

where $\binom{n}{k} = n!/k!(n-k)!$ is the binomial coefficient (see above). Binomial distributions serve as a good approximation for normal distributions. The mean and standard deviation of any binomial distribution could be calculated by very simple formulas: $\mu = np$, $\sigma = \sqrt{npq}$ respectively.

binomial expansion Also called Binomial Theorem. For any given real quantities x and y (fixed or variable) and any natural number n, the following expansion holds:

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} x^0 y^n$$

and $\binom{n}{k}$, $0 \leq k \leq n$ are the binomial coefficients.

binomial series For a real number $\alpha, -\infty < \alpha < \infty$, the series

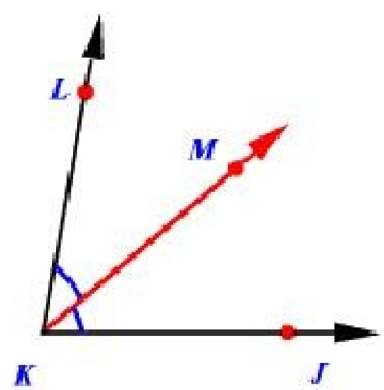
$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n,$$

where

$$\binom{\alpha}{n} = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))/n!$$

is the proper generalization of *binomial coefficients* (see also combinations) to the case of non-integer values of α. The series converges for all $|x| < 1$. In the case when α is a positive integer, the series reduces to finite *binomial expansion*.

bisector of an angle A ray, coming out of the *vertex* of an angle, that divides the given angle into two equal parts. In a triangle, the bisector of any angle is a segment coming out from the vertex, dividing the angle into equal parts. Similarly, bisectors could be defined for angles of any *polygon*.



block of a matrix Let A be some $n \times m$-matrix. We can divide this matrix into parts by separating certain rows and columns. Each part in this case is called a block. Example: The matrix

$$A = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 3 & 4 \\ 5 & -2 & 0 \end{pmatrix}$$

could be divided into four blocks as follows

$$B_1 = (1) \; B_2 = (-1 \; -2) \; B_3 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$$B_4 = \begin{pmatrix} 3 & 4 \\ -2 & 0 \end{pmatrix}.$$

This division is not unique and depends on what we need to do with this matrix. Operations done with blocks of the matrix are called block manipulations.

Boolean algebra The system of rules of operations with logical statements. This algebra involves Boolean variables, that take True and False values (or 1 and 0) and operations of conjunction (”and”), disjunction (”or”) and negation (”not”). Named after George Boole.

boundary For any set in the Euclidean space R^n, $n \geq 1$, the boundary is the set of all *boundary points* (see below). Examples: (1) For the interval $[a, b]$ the boundary consists of two points a and b. (2) For the ball $x^2 + y^2 + z^2 \leq 1$, the boundary is the sphere $x^2 + y^2 + z^2 = 1$.

boundary conditions (1) For ordinary differential equations. If the equation is given on some finite interval $[a, b]$, then the boundary consists of only two points a and b and the conditions are given on both points. The most general conditions could be written as $\alpha_1 y(0) + \alpha_2 y'(0) = 0$ and $\beta_1 y(1) + \beta_2 y(1) = 0$.
(2) For partial differential equations. If the equation is given on some region D, then boundary is either some *curve* or a *surface*. Accordingly, the boundary conditions would be given on that boundary.

boundary curve A curve that makes the *boundary* of some plane region.

boundary point For a set D in the *Euclidean space* R^n, $n \geq 1$, a point $a \in D$ is a boundary point, if any ball with center at that point contains both points in D and outside of D. In case $n = 1$ the ball should be substituted by interval and in case $n = 2$ it should be a circle. Examples: (1) For the interval $[a, b]$ each of the endpoints is a boundary point. (2) For the circle $x^2 + y^2 \leq 1$ the point $(1, 0)$ is a boundary point.

boundary value problem For ordinary differential equations. A differential equation along with *boundary conditions* for solutions. To solve a boundary value problem means to find a solution to the given equation that also satisfies conditions on the boundary. The boundary conditions are chosen to assure that the solution is unique. Example: The equation

$$p(x)y'' - q(x)y + \lambda r(x)y = 0$$

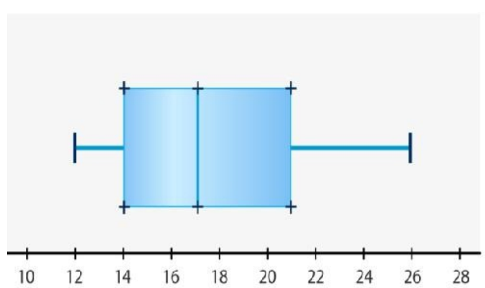
on the interval $[0, 1]$ with be conditions $\alpha_1 y(0) + \alpha_2 y'(0) = 0$ and $\beta_1 y(1) + \beta_2 y(1) = 0$ is called *Sturm-Loiville* boundary value problem.

bounded function A function f, defined on some interval I (finite or infinite), is bounded, if there exists a number $M > 0$, such that $|f(x)| \leq M$ for all values $x \in I$.

bounded sequence A sequence $\{a_n\}$ of real or complex numbers is bounded if there exists a number $M > 0$ such that $|a_n| \leq M$ for all values of $n \geq 1$.

bounded set A subset S of the real line R is bounded, if there is a positive number $M > 0$ such that all elements $x \in S$ satisfy $|x| < M$. If the set S is a subset of R^n, then similar definition applies with the $|x|$ understood as the length of the point $x \in R^n$.

boxplot One of the tools of visualizing *quantitative data*. To construct a boxplot it is necessary first to organize the values in increasing order. Next, finding the median of these values we indicate the center. On the last step we find the two medians of these two halves. These points are the first and third quartiles of the data set. Along with the minimum and maximum values they make-up the *five point summary*. Now, the boxplot consists of a box where these five values are indicated. The form of the boxplot is not standard and could be made both horizontal or vertical. Picture shows one of the possible forms.



braces Or curly brackets.The symbols { }. One of the grouping symbols along with parentheses and brackets. Primarily is used to separate certain numbers and variables to indicate operations to be done first.

brachistochrone For given two points on the plane, brachistochrone is the curve of quickest descent between these points. This curve is not the shortest (which is the straight line) and turns out to be a cycloid.

branches of hyperbola Any non-degenerate hyperbola consists of two separate curves, called branches.

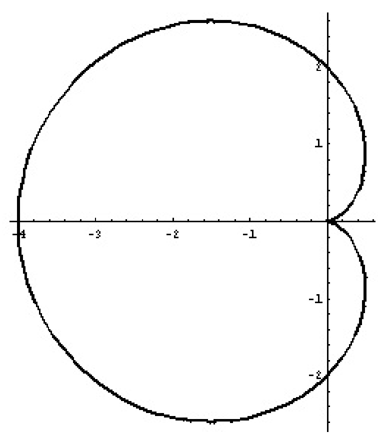
branches of the tangent The graph of the tangent function consists of infinitely many identical curves, called its branches. These branches are separated from each other by the vertical asymptotes $x = \frac{\pi}{2} + \pi n$, n is an arbitrary integer.

brackets The symbols [ ]. One of the *grouping symbols* along with parentheses and braces. Primarily is used to separate certain numbers and variables to indicate operations to be done first.

C

calculus One of the major branches of mathematics, along with algebra and geometry. Calculus itself consists of two main branches: differential and integral calculus. These two branches are related by the Fundamental Theorem of Calculus.

Cantor set Also called the Cantor ternary set.Take the closed interval $[0, 1]$ and remove the middle open interval $(1/3, 2/3)$ from if. The result will be the union of two closed intervals $[0, 1/3]$ and $[2/3, 1]$. From each of these intervals we again remove their middle third open intervals $(1/9, 2/9)$ and $(7/9, 8/9)$ and get the union of four closed intervals $[0, 1/9]$, $[2/9, 1/3]$, $[2/3, 7/9]$, $[8/9, 1]$. If we continue this process indefinitely, the result will be a closed set called Cantor set. This set has many remarkable properties. In particular, it is an infinite set of measure zero and it is a perfect set in the sense that for any point in the set there is an infinite sequence of points from that set that approach that point.



cardioid Any of the plane curves given in polar coordinates by the equations

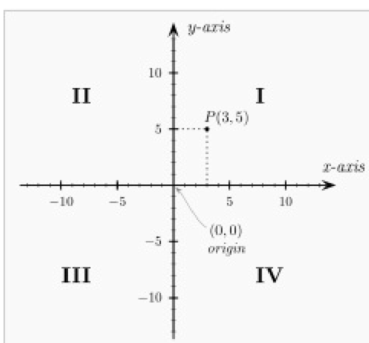
$$r = a(1 \pm \cos\theta)$$

or

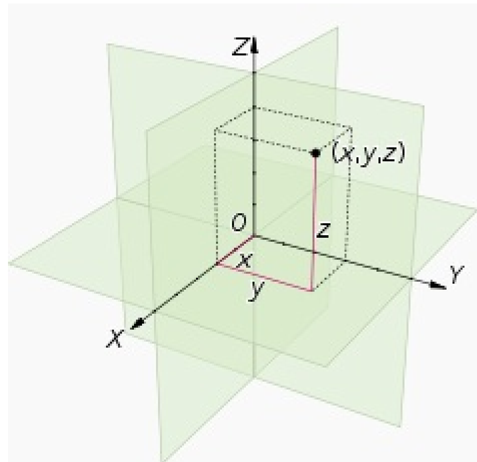
$$r = a(1 \pm \sin\theta$$

Cardioids are special cases of limaçons.

Cartesian coordinate system (1) For points on a plane, also called the rectangular coordinate system. A method of representing points on a plane as *ordered pairs* of numbers and vice versa. To do so, a pair of perpendicular lines (called *axes*) are drawn, that intersect at a point called the *origin*. The horizontal line is called the x-axis and the vertical one – the y-axis. Now, coordinates of any point are determined by drawing perpendicular lines from a point until they reach the axes. The points of these intersections (which represent numbers on corresponding number lines) make up a pair of numbers, called the ordered pair. Conversely, if we have a pair of numbers, then the point corresponding to it could be found by putting these numbers on appropriate axes and drawing two perpendiculars, until they meet at some point.



(2) For points in three dimensional space. To represent points in the space one additional coordinate axis (called the z-axis) is necessary. This allows to represent any point as an ordered triple, just as in the case of the plane. Similarly, any ordered triple will represent a point in the space.



(3) The system works also for spaces of any dimension $n \geq 2$ and even for infinite dimensional spaces. The importance of the Cartesian coordinate system is that it allows to connect geometry with algebra, calculus, and other branches of mathematics. Named after René Descartes.

Cartesian plane A plane, equipped with the *Cartesian coordinate system*.

Cartesian product For any two given sets A and B (of arbitrary nature) the set of all possible ordered pairs (x, y), where x is an element of A and y is an element of B. The notation is $A \times B$. The Cartesian product can be defined for any number of sets and even for infinitely many sets.

Cauchy's mean value theorem Let the functions $f(x)$ and $g(x)$ be continuous on a closed interval $[a, b]$ and be differentiable on the open interval (a, b). Then there exists a point c, $a < c < b$, such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

This theorem is the generalization of the Mean value theorem.

Cauchy-Euler equation Also called *Euler equation*. The equation

$$x^2 u'' + bxu' + cu = 0,$$

where b and c are real numbers. The solutions of this equation depend on the solutions of the algebraic equation $F(r) = r(r-1) + ar + b = 0$.

(1) If the quadratic equation has two real solutions $r_1 \neq r_2$ then the general solution of the Cauchy-Euler equation is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$.

(2) If the quadratic equation has the a repeated solution r then the general solution is given by the formula $y = c_1|x|^r + c_2|x|^r \ln|x|$.

(3) In the case of two complex solutions $r = \lambda \pm i\mu$, the solution of the differential equation comes in the form

$$y = |x|^\lambda[c_1 \cos(\mu \ln|x|) + c_2 \sin(\mu \ln|x|)].$$

Cauchy-Schwarz inequality For any two non-zero sequences of real numbers $\{a_k\}_{k=1}^n$ and $\{b_k\}_{k=1}^n$, the inequality

$$(\sum_{k=1}^n a_k b_k)^2 \leq \sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2$$

holds. The equality holds if and only if $a_k = cb_k$ for some constant c. The same inequality is true even in the case of infinite sequences.

Cayley-Hamilton theorem Let A be a square matrix and let

$$c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_{n-1}\lambda^{n-1} + \lambda^n = 0$$

be its *characteristic equation*. Then the matrix A satisfies the equation

$$c_0 I + c_1 A + c_2 A^2 + \cdots + c_{n-1}A^{n-1} + A^n = 0,$$

where I is the identity matrix.

center of a circle The point that has equal distance from each point of the circle.

center of the distribution A somewhat intuitive notion that is not exactly defined and may mean different thing for different distributions. Most commonly the center refers to the mean of the distribution μ.

central angle An angle, that is formed by two *radii* of a circle.

central limit theorem The most fundamental result in Statistics. Informally, it states that if we form a sampling distribution from a large population then the result will be another distribution that is approximately normally distributed and this new distribution has the same mean as the original distribution. The standard deviations are also related. More precisely:
Suppose we have some distribution with the mean μ and standard deviation σ. If we form a new distribution from the means of simple random samples of size n, then this distribution will be close to a normal distribution with mean μ and standard deviation σ/n. The larger n is, the better is the approximation.

centroid Let $\mathcal{R}$ denote a plate with uniform mass distribution ρ. The center of mass of this plate is called centroid of $\mathcal{R}$. There are different formulas for locating the coordinates of the centroid. If the region $\mathcal{R}$ is bounded by the function $f(x) \geq 0$ and the lines $y = 0$, $x = a$, $x = b$, then the coordinates of the centroid are given by the formulas

$$\overline{x} = \frac{1}{A}\int_a^b x f(x)dx, \quad \overline{y} = \frac{1}{2A}\int_a^b [f(x)]^2 dx,$$

where $A = \int_a^b f(x)dx$ is the area of the region. In the case when $\mathcal{R}$ is bounded by two functions $f(x) \geq g(x)$, the formulas should be adjusted and will include $f(x) - g(x)$ in the formula for $\overline{x}$ and $[f(x)]^2 - [g(x)]^2$ in the formula for $\overline{y}$.

chain rule The formula for calculating the derivative of a composite function. (1) For functions of one variable. Let $F(x) = f(g(x))$ and both $f(x)$ and $g(x)$ are differentiable. Then

$$F'(x) = f'(g(x)) \cdot g'(x).$$

(2) For functions of several variables: Let $z = z(x_1, \cdots, x_m)$ be a differentiable function of m variables and each of these variables are themselves differentiable functions of n variables: $x_1 = x_1(t_1, \cdots, t_n), \cdots, x_m = x_m(t_1, \cdots, t_n)$. Then the following formulas for the partial derivatives hold:

$$\frac{\partial z}{\partial t_1} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \cdots + \frac{\partial z}{\partial x_m}\frac{\partial x_m}{\partial t_1}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial z}{\partial t_n} = \frac{\partial z}{\partial x_1}\frac{\partial x_1}{\partial t_n} + \cdots + \frac{\partial z}{\partial x_m}\frac{\partial x_m}{\partial t_n}$$

change of base For logarithmic function: If a and b are both positive real numbers not equal to 1, and $x > 0$, then

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

change of variable A method used in many situations to solve certain types of equations or evaluate *definite or indefinite integrals*. Example: To solve the equation $x^{1/2} - 5x^{1/4} + 6 = 0$ we make a change of variable $y = x^{1/4}$ to translate to the quadratic equation $y^2 - 5y + 6 = 0$ which has the solutions $y = 2, 3$. Returning to the original variable x we get $x^{1/4} = 2, 3$ or $x = 16, 81$.
For examples with integration see the entry substitution method.

characteristic equation (1) The equation $p(x) = 0$, where p is the *characteristic polynomial* of a given square matrix. If the matrix has the size $n \times n$, then the polynomial has degree n and, by the Fundamental Theorem of Algebra, has exactly n roots (if counted with multiplicities). The roots of the equation may be both real or complex and some of them might be repeated. See also eigenvalues, eigenvectors, and eigenspace.
(2) For a linear homogeneous differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + c_{n-1} y' + c_n y = 0$$

the corresponding polynomial equation

$$a_0 r^n + a_1 r^{n-1} + \cdots + c_{n-1} r + c_n = 0.$$

The solutions of this equation are called characteristic roots. For the importance and the use of these roots see the article *linear ordinary differential equations*.

characteristic polynomial (1) Let A be an $n \times n$ *matrix* and I is the identity matrix of the same size. The *determinant* of the matrix $A - \lambda I$ is a polynomial, called the characteristic polynomial of the matrix A. This polynomial has degree n and, by the *Fundamental Theorem of Algebra*, has exactly n roots (if counted with multiplicities). This roots are called the *eigenvalues* of A. Example: For the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & -4 & 1 \end{pmatrix}$$

we have

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & -4 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda)(3-\lambda)(1-\lambda),$$

which has three real and distinct roots: 1,2,3. See also *eigenvalues* and *eigenvectors*.
(2) For a *linear homogeneous differential equation*

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + c_{n-1} y' + c_n y = 0$$

the corresponding polynomial

$$p(r) = a_0 r^n + a_1 r^{n-1} + \cdots + c_{n-1} r + c_n$$

of degree n with respect to the variable r is its characteristic polynomial. The roots (zeros) of the polynomial are the characteristic roots of the polynomial.

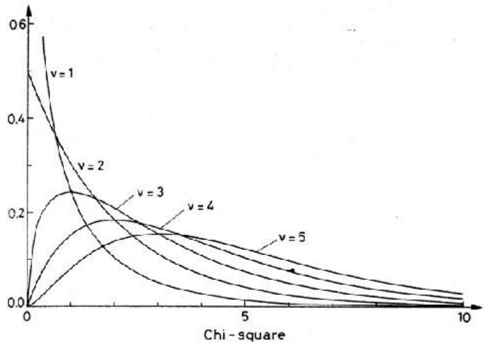
Chebyshev equation The differential equation

$$(1 - x^2) y'' - x y' + \alpha y = 0.$$

Solutions of this equation when α is a non-negative integer n, are called Chebyshev polynomials. These polynomials could be written in many different forms. The simplest and most used form is

$$T_n(x) = \frac{(x + \sqrt{x^2 - 1})^n + (x + \sqrt{x^2 - 1})^{-n}}{2}.$$

chi-square distribution The distribution of



variance. There are multiple definitions for this distribution and the following is one of the simplest:

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2},$$

where n is the sample size, s^2 is the sample variance and σ^2 is the population variance. Because variance is positive the curve of the distribution is always on the right side. As the graph shows, the distribution becomes more symmetric with the increasing size of the sample.

chord For any given *curve*, a straight line segment, that connects any two points on the curve. Most commonly is used for the circle.

circle A plane geometric figure with the property that there exists a point (called *center of the circle*) and a positive number r (called the *radius*), such that the distance from each point of that figure to the center is equal to r.

circular cylinder A cylinder with a circle base.

circumference The perimeter of a *circle*.

Clairaut's theorem Also known as the theorem of the change of order of partial differentiation. If the function $f(x_1, x_2, \ldots, x_n)$ has continuous second partial derivatives at some point $(a_1, a_2, ..., a_n)$ in its domain, then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a_1, a_2, ..., a_n) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a_1, a_2, ..., a_n)$$

for any combination of indices $1 \le i, j \le n$.

closed curve A curve that has no end points. If a plane curve is given by the parametric equations $x = x(t),\ y = y(t),\ t \in [a, b]$, then the curve is closed if $x(a) = x(b)$ and $y(a) = y(b)$.

closed interval Intervals of the form $[a, b]$, where both end-points are finite and included.

closed set A set that contains all of its boundary points. For example, the interval $[0, 1]$ is a closed set because it contains both of its boundary points 0 and 1, while the interval $(0, 1)$ is not closed. The set $\{(x, y)|x^2 + y^2 \le 1\}$ is a closed set on the plane but the set $\{(x, y)|x^2+y^2 < 1\}$ is not a closed set because it does not contain its boundary.

closed surface In the simplest form, a closed surface is the *boundary* of a bounded solid domain. The general definition, given in advanced calculus courses, requires notions from set theory and topology.

closed under A set of numbers (or, more generally, some objects) is closed under certain operation, if the result of that operation also belongs to that set. Examples. (1) The set of all *integers* is closed under both operations of addition and multiplication, because the result is also an integer. (2) The set of real numbers is closed under addition, subtraction, multiplication, and division (except by zero), but it is not closed under the operation of square root extraction. (3) The set of all square matrices is closed under addition, scalar multiplication and also matrix multiplication.

coefficient A word most commonly used to indicate the non-variable factor of a term of a polynomial. See *coefficients of a polynomial.* This term is also used to describe the numeric components in functions other than polynomials.

coefficient matrix For a system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

the $m \times n$ matrix that consists of all coefficients of the system (but not the constant parts on the right sides):

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

When a column of the constant terms is attached, the matrix is called augmented.

coefficients of a polynomial In the polynomial of arbitrary degree

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

the constants $a_0,\ a_1, \cdots, a_n$.

coefficients of power series In the power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

the numeric constants $a_n,\ n = 0, 1, 2, 3 \cdots$.

cofactor For a given $n \times n$ matrix A, the (i, j)th cofactor is the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and jth column of A, multiplied by $(-1)^{i+j}$. Example: If

$$A = \begin{pmatrix} -4 & 4 & 1 \\ 0 & -1 & 3 \\ 2 & 5 & 2 \end{pmatrix}$$

then the cofactor corresponding to the element a_{23} (which is 3) is

$$(-1)^{2+3} \det \begin{pmatrix} -4 & 4 \\ 2 & 5 \end{pmatrix} = (-1)(-20-8) = 28.$$

See also minors of a matrix.

cofactor matrix Another name for the adjoint matrix.

cofunction The common name of trigonometric functions $\cos x,\ \cot x,\ \csc x$.

cofunction identities In trigonometry, the identities

$$\cos(\frac{\pi}{2} - \theta) = \sin\theta, \quad \sin(\frac{\pi}{2} - \theta) = \cos\theta;$$

$$\cot(\frac{\pi}{2} - \theta) = \tan\theta, \quad \tan(\frac{\pi}{2} - \theta) = \cot\theta;$$

$$\csc(\frac{\pi}{2} - \theta) = \sec\theta, \quad \sec(\frac{\pi}{2} - \theta) = \csc\theta,$$

which are true for any real values of θ where the functions are defined.

column matrix A *matrix*, that consists of one column, or an $m \times 1$ matrix. It is the same as a *vector*, written in column form. Example:

$$\begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}.$$

column space Let A be an $m \times n$ matrix. The subspace of the space R^m that is spanned by the *column-vectors* of that matrix is called the column space of A.

column vector A vector written in column form. The same as column matrix above.

combination For the given set S of n elements, combination of k elements is a subset of S, that contains exactly k elements. At least one element should be different and two sets with the same elements are considered the same. For example, the subsets $\{1, 2, 3\}$ and $\{2, 1, 3\}$ are the same. The number of combinations of k elements out of n is given by the formula

$$C_k^n = \binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $0 \le k \le n$. See also binomial coefficients and permutation.

common denominator If two or more fractions have the same denominator, then that denominator is called the common denominator (of the fractions). If the denominators are different, the fractions could be changed to an equivalent form with a common denominator. See least common denominator for details.

common factors In any *factorization* of two or more natural numbers, any factors that are the same for all of these numbers. Example: The numbers $24 = 2^3 \cdot 3$ and $30 = 2 \cdot 3 \cdot 5$ have three common factors, 2, 3, and 6.

common logarithms Logarithmic function with base 10. The base of common logarithms is often not written but is understood: $\log x = \log_{10} x$.

common multiple For two or more integers, any integer that is divisible by all of them. The number 30 is a common multiple of 10 and 6, because $30 = 10 \cdot 3$ and $30 = 6 \cdot 5$.

commutative property for addition For any given real or complex numbers a and b, $a + b = b + a$, or, the order of the *addends* does not change the sum.

commutative property for multiplication For any given real or complex numbers a and b, $a \cdot b = b \cdot a$, or, the order of the *factors* does not change the product.

commuting matrices Two square matrices A and B such that $A \cdot B = B \cdot A$.

comparison properties of the integral For integrable functions $f(x)$ and $g(x)$ of one real variable defined on $[a, b]$:
(1) If $f(x) \geq 0$ then $\int_a^b f(x)dx \geq 0$
(2) If $f(x) \geq g(x)$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
(3) If $m \leq f(x) \leq M$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

comparison theorem for integrals Let f and g be two functions defined on $[a, \infty)$, with the property $f(x) \geq g(x)$ for $x \geq a$.
(1) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$ is also convergent.
(2) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$ is also divergent.

comparison test for series Let $\{a_n\}$ and $\{b_n\}$ be two numeric sequences.
(1) If $0 \leq a_n \leq b_n$ for all n and $\sum_{n=1}^\infty b_n$ is convergent, then $\sum_{n=1}^\infty a_n$ is convergent.
(2) If $a_n \geq b_n \geq 0$ for all n and $\sum_{n=1}^\infty b_n$ is divergent, then $\sum_{n=1}^\infty a_n$ is divergent.

complement of a set Let S be a space of sets or a set of some elements and assume A is an element of that space. The complement of A is the set $\bar{A}$, that has no intersection with A ($A \cap \bar{A} = \emptyset$), and their union "fills" the space, $A \cup \bar{A} = S$.

complementary angles Two angles are complementary, if the sum of their measures is $90°$ (in degree measure) or $\pi/2$ (in radian measure).

completing the square If a quadratic trinomial is not a perfect square, it is still possible to represent it as a sum of a perfect square and some constant. This procedure is called completing the square. If the trinomial is $ax^2 + bx + c$, then we isolate the first two terms and write

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x\right) + c$$

and notice that in order to make the expression inside the parenthesis a perfect square we are missing the square of half of the linear term's coefficient. Adding and subtracting that number we get

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right) + c$$
$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a} + c$$
$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c.$$

Examples:
(1) $x^2 - 6x + 7 = (x^2 - 6x + 9) - 9 + 7 = (x-3)^2 - 2$.
(2) $3x^2 + 7x - 5 = 3(x^2 + \frac{7}{3}x) - 5 = 3(x^2 + \frac{7}{3}x + \frac{49}{36} - \frac{49}{36}) - 5 = 3(x^2 + \frac{7}{3}x + \frac{49}{36}) - \frac{49}{12} - 5 = 3(x + \frac{7}{6})^2 - \frac{109}{12}$.

complex conjugates Complex numbers with the same real parts and opposite imaginary parts: $a + ib$ and $a - ib$. The complex conjugate of the number z is denoted by $\overline{z}$. A very important property of complex conjugates is that their product is always a positive number or zero (if the number is zero itself):

$$(a + ib)(a - ib) = a^2 - i^2b^2 = a^2 + b^2 \geq 0.$$

This fact is used in the process of the division of two complex numbers.

complex exponents The complex exponent for the base e is defined by the use of Euler's formula. If $z = x + iy$ is some complex number, then

$$e^z = e^x(\cos y + i \sin y).$$

For any other positive base $a \neq 1$ the complex exponent could be defined similarly with the use of the identity $a = e^{\ln a}$.

complex fraction A fraction, where the numerator, or denominator or both are fractions themselves. Examples:

$$\frac{\frac{2}{3}}{\frac{7}{13}}, \qquad \frac{\frac{x}{2x+1}}{\frac{x^2-1}{x-5}}.$$

complex numbers Numbers of the form $z = x + iy$, where x and y are *real numbers* and $i = \sqrt{-1}$, is the *imaginary unit*. x is called the real part of the complex number and y is the imaginary part. Two complex numbers are equal, if their real and imaginary parts are identical: $a + ib = c + id$ if and only if $a = c$ and $b = d$. Each complex number can be represented graphically as a point on the plane, where the real part is identified with the x-coordinate and the imaginary part with the y-coordinate.

Complex numbers also have a *trigonometric* representation of the form $z = r(\cos\theta + i\sin\theta)$, where $r = |z| = \sqrt{x^2 + y^2}$ is the *modulus* of the number and the angle θ, (called the argument of z) is determined from the equation $\tan\theta = y/x$.

For operations with complex numbers see addition and subtraction of complex numbers, division of complex numbers, multiplication of complex numbers, DeMoivre's theorem.

complex plane A plane equivalent to the Cartesian plane with one significant difference: the points on the plane are not viewed as *ordered pairs* (x, y), but rather as points with *real* and *imaginary parts* coinciding with these same values $x,\ y$. Hence, arithmetic operations can be done with the points on the complex plane according to the rules of operations with complex numbers.

component function If $\mathbf{r}$ is a vector-function in the three dimensional space, then it could be written in the form

$$\mathbf{r}(t) = (f(t), g(t), h(t)).$$

The functions $f,\ g,\ h$ are called component functions. The same notion is valid for two dimensional vector-functions also.

components of a vector Let $\mathbf{v}$ be some vector in a vector space V. Then it could be written in its component form $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ and the *scalars* $v_1, v_2, \cdots, v_n$ are the components of this vector. Components depend on the basis of the space V and vary with change of basis. Example: the vector $\mathbf{v} = (2, 3)$ in the standard basis $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$ of R^2, in the nonstandard basis $\mathbf{e}_1' = (-1, 0)$, $\mathbf{e}_2' = (0, -1)$ will have components $(-2, -3)$.

composite function See *composition of functions*.

composite number A natural number, that could be written as a product of two other natural numbers different from 1 or the number itself. If it is not possible, the number is called *prime*.

composition of functions (1) For functions of one variable. Let $f(x)$ and $g(x)$ be two functions and assume that the *range* of g is contained in the *domain* of f. Then the composition of f with g is defined to be the function $h(x) = f(g(x))$. The notation $h = f \circ g$ is commonly used. The operation of composition is not *commutative*. Example: Let $f(x) = x^2 + 1$ and $g(x) = 2x - 1$. Then $(f \circ g)(x) = 4x^2 - 4x + 1$ and $(g \circ f)(x) = 2x^2 + 1$.

(2) For functions of several variables. If $f = f(x_1, \cdots, x_m)$ and there are m functions $g_k(x_1, \cdots, x_n),\ 1 \leq k \leq m$, of n variables, then the composition function can be defined by the formula

$$h(x_1, \cdots, x_n) =$$

$$f(g_1(x_1, \cdots, x_n), \cdots, g_m(x_1, \cdots, x_n)).$$

In this definition we have to assume also that the combined ranges of the functions g_k are in the domain of f. In the several variable case the composition is also not commutative.

composition of linear transformations Let $T_1 : V_1 \to V_2$ be a linear transformation between two linear vector spaces V_1 and V_2. Assume also, that another linear transformation T_2 between V_2 and some other linear vector space V_3 is defined in such a way, that its domain is contained in the range of the transformation T_1. Then the composition transformation

$T(\mathbf{v}) = T_2(T_1(\mathbf{v})) = (T_2 \circ T_1)(\mathbf{v})$ is a transformation between the spaces V_1 and V_3. The composition operation is not commutative: $T_1 \circ T_2 \neq T_2 \circ T_1$ in general.

compound interest When a given amount of money is invested and the interest is paid not only on the principal amount but also on the interest earned on the principal, it is called compound interest. If the principal amount is denoted by P, the interest rate by r and interest payments are made n times a year, then the formula for the amount A after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}.$$

See also continuously compounded interest.

concave function A differentiable function on some interval is concave down if its derivative is a decreasing function. Geometrically this property means that the tangent line to the graph of the function at any point is above the graph. Similarly, a differentiable function on some interval is concave up if its derivative is an increasing function. Geometrically this property means that the tangent line to the graph of the function at any point is below the graph. Concave up functions are also called *convex*.

concavity The property of being concave.

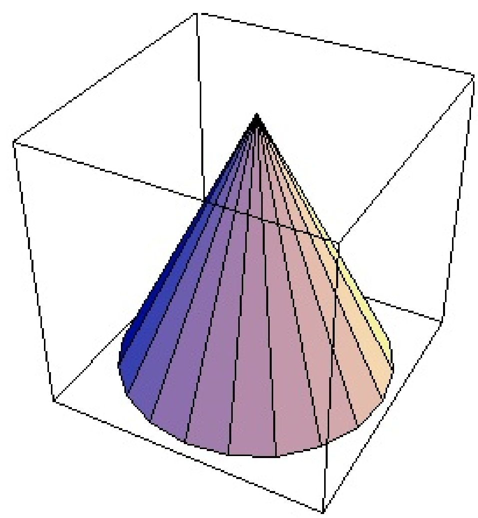
concavity test Let $f(x)$ be a twice *differentiable* function on some interval I. (1) If $f'(x) > 0$ for all values of x on the interval, then the function is *concave up*. (2) If $f'(x) < 0$ for all values of x on the interval, then the function is *concave down*.

conditional probability The probability of some event A under the assumption that event B happens too. The notation is $P(A|B)$. If two events A, B are not mutually exclusive, then the probability that both events happen is given by the general multiplication formula

$$P(A \cap B) = P(A)P(B|A).$$

conditional statement A logical statement of the form "If A, then B", written also in the short form $A \Rightarrow B$. Examples include: "If x is real, then x^2 is positive", "If a function is differentiable, then it is continuous". To each conditional statement $A \Rightarrow B$, there exist three other statements.
(1) Converse: $B \Rightarrow A$, meaning "If B, then A". If the direct statement is true, the converse may or may not be true.
(2) Inverse: not $A \Rightarrow$ not B, meaning "If A is not true, then B is not true". As in the case of converse, the inverse may or may not be true.
(3) Contrapositive: not $B \Rightarrow$ not A, meaning "If B is not true, then A is not true". Direct and contrapositive statements are equivalent in the sense that either they both are true or both are false.
See also biconditional statement, logical contrapositive.

conditionally convergent series A series that converges but the series formed by the absolute values of its terms does not. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent because the series formed by its absolute values is the divergent harmonic series $\sum_{n=1}^{\infty} 1/n$. If a series is conditionally convergent, then its terms could be rearranged so that the sum equals any given number.



cone A three dimensional geometric object generated by a line (called the generator) rotating about a fixed point on the line called the apex. Both the surface and the solid generated this way are called cone. The picture above shows one half of a right circular cone.

confidence interval In statistics. An *interval* of real values where we expect to have the "majority" of values of some distribution concentrated. The two endpoints of that interval are given by the expression $Estimate \pm Margin\, of\, Error$. For example, for the distribution of proportions, this interval could be given by the expression

$$\hat{p} \pm z^* SE(\hat{p}),$$

where $\hat{p}$ is the estimated proportion of the sample data, $SE(\hat{p})$ is the *standard error* of that sample data, and z^* is the critical value corresponding to the given *confidence level*. The length of the interval depends on the size of the data set and on the desired confidence level.

confidence level A percentage we chose to be confident of the accuracy of statistical estimates that come from sample data. The most common values are 90%, 95%, and 99%. The confidence level is associated with *critical values.*

congruence An equivalence relation between two or more sets. Geometric figures are called congruent if one figure (plane or solid) could be moved and/or rotated and/or reflected to coincide with the other figure. Congruence is different from equality in the sense that these two figures are not the same because originally they occupied different places on the plane or in space while equal figures are supposed to have all components of the same size and occupy the same space.

conic section The result of cutting a double cone by a plane. Depending on the position of the plane the result could be a circle, ellipse, parabola, hyperbola or a degenerate conic section: a point, a pair of intersecting lines, or the empty set. An alternative geometric definition could be given using eccentricity.

Algebraically, any conic section could be described as a solution of a quadratic equation in two variables: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, C, D, E, F are real constants. See the entries *ellipse, hyperbola, parabola* for the details and additional information regarding *foci, directrix, axes* etc.

conjugates The expressions $a + b$ and $a - b$ are called conjugates or conjugate pairs. The term most often refers to complex conjugates. It is also used for expressions of the form $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$. Using these conjugates is important when rationalizing denominators containing these expressions.

conjugate axis See hyperbola.

connected region If any two points of a region can be connected by a path (curve) that lies completely inside that region, then the region is called connected.

conservative vector field A vector field $\mathbf{F}$ is called conservative, if there exists a scalar function f such that its gradient vector field coincides with $\mathbf{F}$: $\mathbf{F}{=}\bigtriangledown f$.

consistent linear system A system of equations which has at least one solution. When the system has no solution, it is called *inconsistent.*

constant A quantity that does not change, usually some type of number.

constant multiple rule See differentiation rules.

constraint A term used as a synonym of restriction. The constraints could be on an independent variable as well as on a function itself. Example: Find the maximum value of the function $f(x, y) = 2x^2 - 3xy + y^2$ under the constraint $x^2 + y^2 = 1$.

continued fraction An expression of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cdots}}.$$

Continued fraction might be finite or continue indefinitely.

continued fraction expansion Representation of a number or a function as a finite or infinite continued fraction. Example:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}.$$

continuity of a function The function $f(x)$, defined on some interval $[a, b]$ is continuous at a point $c \in (a, b)$, if

$$\lim_{x \to c} f(x) = f(c).$$

continuity on an interval The function is continuous on an interval, if it is continuous at each point of that interval. If the function is defined on a closed interval $[a, b]$ then for the continuity at the end-points a and b see the entries *continuity from the right* and *continuity from the left* respectively below.

continuity from the left The function $f(x)$ is continuous from the left at some point c, if at that point the *left-hand limit* exists and is equal to the value of the function at that point: $\lim_{x \to c^-} f(x) = f(c)$.

continuity from the right The function $f(x)$ is continuous from the right at some point c, if at that point the *right-hand limit* exists and is equal to the value of the function at that point: $\lim_{x \to c^+} f(x) = f(c)$.

continuous function See *continuity of a function.*

continuously compounded interest When a given amount of money is invested and the interest is paid not only on principal amount but also on interest earned on principal, and the number of payments is unlimited, then it is called continuously compounded interest. If the principal amount is P and the interest rate is r, then the amount of money after t years is given by the formula $A = Pe^{rt}$. See also compound interest.

contraction operator Or contraction transformation. An operator applied to a vector does not change its direction but makes its *magnitude* smaller. The operator is given by the formula $T\mathbf{x} = k\mathbf{x}$, where $0 \leq k \leq 1$. In the case $k \geq 1$ the transformation is called a *dilation operator.* For a more general case when not only the magnitude but also the direction is changed, see *expansion operator.*

contrapositive See conditional statement.

convenience sampling A type of sampling when the samples are taken based on the convenience of the person taking them. This method is not considered scientifically reliable or unbiased.

convergence For specific definitions see *convergent integrals, convergent sequences, convergent series.* See also absolutely convergent series, conditionally convergent series.

convergent integral A definite integral with finite value. For proper integrals this means that the Riemann sums have finite limit. The same applies also to improper integrals.

convergent sequence A numeric sequence that approaches a finite value. The formal definition is: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of (real or complex) numbers. The sequence converges to some number A, if for any given $\varepsilon > 0$ there exists an $N > 0$, such that $|a_n - A| < \varepsilon$ whenever $n > N$. In an equivalent notation, $\lim_{n \to \infty} a_n = A$.
Example: The sequence $\{1 + 1/n\}$ converges to 1 as $n \to \infty$.

convergent series A numeric series is convergent if the *partial sums* of that series $S_n = \sum_{k=1}^{n} a_k$ form a convergent sequence. More precisely, if for any given $\varepsilon > 0$ there exists a number N such that $|S_n - L| < \varepsilon$ for any $n > N$, we say that the series converges to L.
Example: The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent for any value $p > 1$. The same exact definition is valid also for functional series, such as power series or Fourier series.

converging solutions A term commonly used to describe approximate solutions that have the property of approaching the exact solution during a certain limiting process. As an example see Newton's method for conditions when approximate solutions approach exact solution.

converse See conditional statement.

convex function A differentiable function on some interval is convex, if its derivative is an increasing function. Geometrically this property means that the tangent line to the graph of the function at every point is always below the graph. Convex functions also are called *concave up.*

convex set A set in *Euclidean* space with the property that for any two points points in that set, the line connecting those points lies completely in the set.

convolution integral For two functions defined on the real axis R, the formal integral

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x - t)dt.$$

Convolution is commutative in the sense that $f * g = g * f$. Another important property of convolution is that the Laplace transform translates convolution into multiplication: Let $\mathcal{L}f(x)$ and $\mathcal{L}g(x)$ be Laplace transforms of f and g respectively. Then $\mathcal{L}(f * g)(x) = \mathcal{L}f(x) \cdot \mathcal{L}g(x)$.

coordinate axes See Cartesian coordinate system.

coordinate matrix If S=$\{\mathbf{v}_1, v_2, ..., v_n\}$ is a basis in some vector space V then any vector $\mathbf{v}$ from V could be written as

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n.$$

The $n \times 1$ column-matrix that consists of the *scalars* $c_1, c_2, ..., c_n$ is called the coordinate matrix of the vector $\mathbf{v}$ with respect to the basis S.

coordinate plane The plane that is determined by any pair of *coordinate axes.* See *Cartesian coordinate system.*

coordinate vector The same as *coordinate matrix*, only written in the form of a $1 \times n$ row-matrix.

coordinate system Any of the methods to identify points on the plane or in space with an *ordered pair* or an *ordered triple.* See the coordinate systems *Cartesian, cylindrical, polar, rectangular, spherical.*

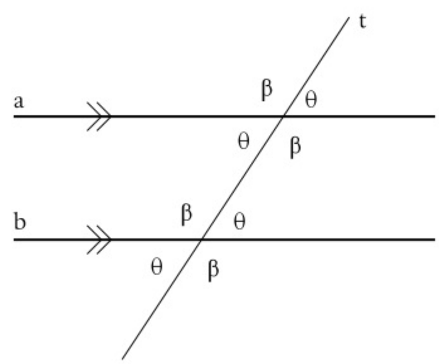
coordinates In any *coordinate system* the ordered pairs or triples corresponding to a point. The same point may have different coordinates in different coordinate systems. Example: The point with coordinates $(-2, 2\sqrt{3})$ in the *Cartesian* system has coordinates $(4, 2\pi/3)$ in the *polar* system.

coplanar vectors Two or more vectors in the *Euclidean space* R^n, that lie on the same plane.

correlation coefficient Suppose we have data in the form of ordered pairs (x, y) and there is some kind of relationship between x and y-variables. Correlation coefficient measures the strength of that relationship in numerical terms. To calculate that coefficient, usually denoted by r, we first normalize the x and y-values by finding their z-scores denoted by z_x and z_y respectively. The sum

$$r = \frac{\sum z_x z_y}{n - 1}$$

is the correlation coefficient. It is defined in a way that $-1 \leq r \leq 1$. A positive r indicates positive correlation and a negative r indicates negative correlation. The closer r is to 1 or -1, the stronger is the linear relationship between the x and y-values.



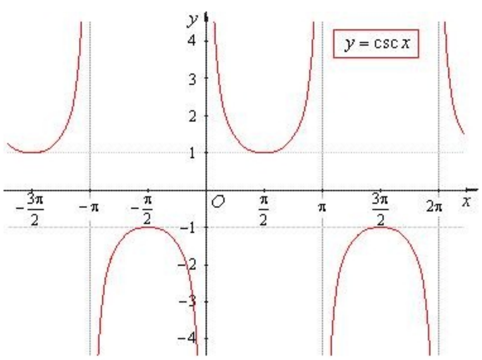
corresponding angles Corresponding angles are formed when a line crosses two coplanar lines. The corresponding angles are not necessarily congruent, but they are if the coplanar lines are also parallel.
In the figure above the lines a and b are parallel and the line t is the transversal. Congruent angles are marked accordingly.

cosecant function One of the six trigonometric functions. Geometrically, the cosecant of an angle in a right triangle is the ratio of the *hypotenuse* of the triangle to the *opposite side.* The cosecant could also be defined as the reciprocal of the sine function. The function $\csc x$ could be extended to all real values ex-

actly as the $\sin x$ function is extended. The domain of $\csc x$ is all real values, except $x = \pi n$, n any integer, and the range is $(-\infty, -1] \cup [1, \infty)$. $\csc x$ is 2π-periodic.
The cosecant function is related to other trigonometric functions by many identities, the most important of these are $\csc x = 1/\sin x$, $1 + \cot^2 x = \csc^2 x$. The derivative and integral of this function are given by the formulas

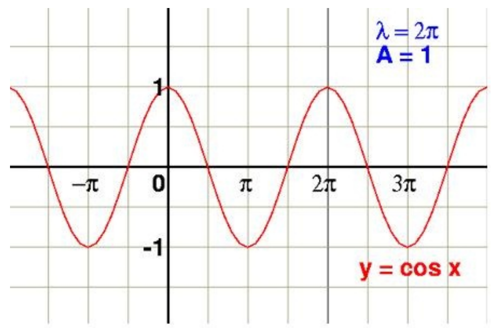
$$\frac{d}{dx} \csc x = -\csc x \cot x,$$

$$\int \csc x dx = \ln|\csc x - \cot x| + C.$$

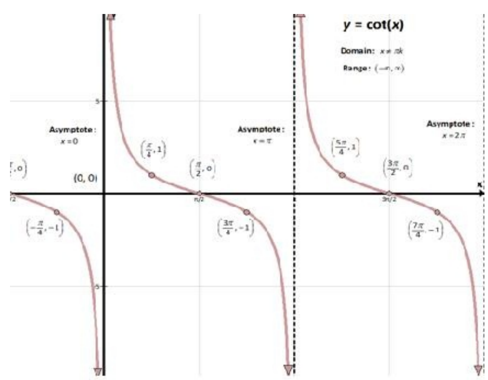


cosine function One of the six trigonometric functions. Geometrically, the cosine of an angle in a right triangle is the ratio of the *adjacent side* to the *hypotenuse* of the triangle. A more general approach to extend the $\cos x$ function for any real number x is as follows: Let $P = (a, b)$ be any point on the plane other than the origin and θ is the angle formed by the x-axes and the terminal side, connecting the origin and P. Then $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$. Next, after establishing one-to-one correspondence between angles and real numbers, we can have the cosine function defined for all real numbers. The range of $\cos x$ is $[-1, 1]$ and it is 2π-periodic.
The cosine function is related to other trigonometric functions by various identities. The most important is the *Pythagorean identity* $\sin^2 x + \cos^2 x = 1$. The derivative and indefinite integral of this function are:

$$\frac{d}{dx}(\cos x) = -\sin x, \qquad \int \cos x dx = \sin x + C.$$



cost function The total cost to a company to produce x units of certain product. Usually denoted by $C(x)$. The derivative of this function is called the marginal cost function. See also average cost function, revenue function..



cotangent function One of the six trigonometric functions. Geometrically, the cotangent of an angle in a right triangle is the ratio of the *adjacent side* of the triangle to the *opposite side*. It could also be defined as the reciprocal of the tangent function. The function $\cot x$ could be extended as a function of all real numbers except for $x = \pi n$, n any integer, and the range is all of R. The $\cot x$ function is π-periodic. The cotangent function is related to the other trigonometric functions by various identities. The most important of these are the identities $\cot x = \cos x/\sin x$, $\cot x = 1/\tan x$ and a version of the Pythagorean identity $1+\cot^2 x = \csc^2 x$. The derivative and integral of this function are given by the formulas

$$\frac{d}{dx}\cot x = -csc^2 x, \int \cot x dx = \ln|\sin x| + C.$$

coterminal angles Two angles in *standard position* which have the same *terminal side* (see the entry angle for explanation of terms above). Any two coterminal angles differ in size by an integer multiple of 360° (in degree measure) or 2π (in radian measure).

counting numbers Another name for *natural* numbers.

Cramer's rule One of the methods of solving systems of linear algebraic equations which involves the use of determinants.
For the system of n equations with n unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

the solution is given by the formulas $x_k = \det(A_k)/\det(A),\ 1 \le k \le n$. Here A is the matrix of the given system of equations and the matrices A_k are found from A by removing the kth column and substituting it by the column of constants $[b_1,\ b_2, \cdots, b_n]$. This formula works if and only if the determinant $\det(A) \neq 0$.
Example: Solve the system

$$x_1 + \qquad +x_n = 6$$

$$-3x_1 + 4x_2 + 6x_n = 30$$

$$-x_1 - 2x_2 + 3x_n = 8$$

The four determinants for this system are:

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{vmatrix},\ |A_1| = \begin{vmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{vmatrix}$$

$$|A_2| = \begin{vmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{vmatrix},\ |A_3| = \begin{vmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{vmatrix}$$

Calculating these determinants, we get $|A| = 44,\ |A_1| = -40,\ |A_2| = 72,\ |A_3| = 152$. Now, Cramer's rule gives the solution $(-12/11, 18/11, 38/11)$.

critical point Also called critical number or value. For a continuous function $f(x)$ on some interval I, a point $c \in I$ is critical, if $f'(c) = 0$ or $f'(c)$ does not exist. Critical points are important because if a function has a local *extremum* at some point in its domain, then that point is a critical point. The opposite is not always true. Examples: (1) For the function $f(x) = x - 2\sin x$ on $[0, 2\pi]$ the critical points are $x = \pi/3, x = 5\pi/3$. The first one is the local minimum and the second one the local maximum of the function on the given interval. (2) For the function

$$f(x) = \begin{cases} 2x & \text{if} \quad x \ge 0 \\ x & \text{if} \quad x < 0 \end{cases}$$

the only critical point is $x = 0$ but at that point the function has neither a minimum nor a maximum value.
The notion of a critical point is also extended to functions of several variables. A point (c_1, c_2, c_3) is a critical point of a function $f(x, y, z)$ of three variables if either all first partial derivatives are equal to zero or some of the first partial derivatives do not exist at that point. As in the case of the functions of one variable, here also the local maximum and minimum values are possible at critical points only, but the opposite is not true: not all critical points are local minimum or maximum points for the function. These kind of points are called *saddle points.*

critical value In statistics. A numeric value associated with any *distribution* that depends on the type of distribution and the confidence level desired. For the standard normal distribution the critical values corresponding to the most common confidence levels 90%, 95%, 99% are $z^* = 1.165, 1.96, 2.576$ respectively.

cross product For two *vectors* in three dimensional space only. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors in R^3. Then their cross product is defined to be another vector from the same space, given by the formula

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

This vector could be expressed as a *determinant*:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

The cross product is not commutative: $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

cubic equation An equation of the form $p(x) = 0$, where p is a *cubic polynomial*. By the *Fundamental Theorem of Algebra*, any cubic equation has exactly three solutions counting multiplicities of zeros. Similar to *quadratic equations*, any cubic equation could be solved by a formula involving radicals, however this formula is very complicated and difficult to use. Here is a short demonstration of that formula and its deduction.
To solve the general cubic equation

$$\alpha x^3 + \beta x^2 + \gamma x + \delta = 0, \quad \alpha \neq 0, \ (1)$$

we divide it by α and get a monic equation (equation with leading coefficient equal to 1)

$$x^3 + ax^2 + bx + c = 0. \qquad (2)$$

Denoting $x = y - a/3$ we reduce this equation to a simpler equation of the form

$$y^3 + py + q = 0, \qquad (3)$$

where $p = b - a^2/3, \quad q = c - ab/3 + 2a^3/27$. Let r denote one of two complex cubic *roots of unity*: $r = -1/2 + \sqrt{3}i/2$ or $r = -1/2 - \sqrt{3}i/2$ and let $\Delta = \sqrt{-(27q^2 + 4p^3)}$, $A = (-27q + 3i\sqrt{3}\Delta)/2$, $B = (-27q - 3i\sqrt{3}\Delta)/2$. The choice of the two possible square roots for Δ and three possible cubic roots of A and B must be made so that $(AB)^{1/3} = -3p$. With this notations, the three roots of equation (3) are given by the following formulas:

$$y_1 = \frac{A^{1/3} + B^{1/3}}{3}, \ y_2 = rA^{1/3} + r^2B^{1/3},$$

$$y_3 = r^2A^{1/3} + rB^{1/3}.$$

To find the solutions of equation (2), we need to make the back substitution $x = y - a/3$. The solutions of equation (1) are obviously the same as that of equation (2). Note also, that at least one of the solutions of the equations (1)-(3) is always real.
Example: For the equation $x^3 - 8x - 3 = 0$ the solutions are 3, $\frac{-3+\sqrt{5}}{2}$, $\frac{-3-\sqrt{5}}{2}$.

cubic polynomial A *polynomial* of the third degree $p(x) = ax^3 + bx^2 + cx + d$. In the particular case, the function $f(x) = ax^3$ is called the cubic function.

cubic root Formally, the inverse of the cubic function. A given number a is the cubic root of another number, b, if $a^3 = b$. The notation is $a = \sqrt[3]{b}$.

curl of a vector field Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field in R^3 and assume that all partial derivatives of the components P, Q, R exist. Then the vector function

$$curl\mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i}$$

$$+ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

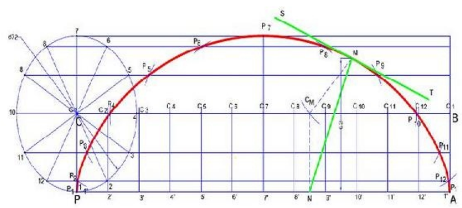
is the curl of the vector field. See also *divergence of a vector field* and *gradient vector field*.

curvature Let C be a *smooth curve* defined by some *vector function* $\mathbf{r} = r(t)$ and let $s = s(t)$ be that curve's *arc length*. If $\mathbf{T} = r'/|r'|$ is the unit tangent vector to the curve then the curvature of the curve is given by the formula

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|.$$

curve (1) A plane curve is a set of all ordered pairs $(f(t), g(t))$, where f and g are *continuous functions* defined on some interval $I = [\alpha, \beta]$. If no points repeat (i.e., if $(f(t_1), g(t_1)) \neq (f(t_2), g(t_2))$ for two values $t_1 \neq t_2$), then the curve is called simple. If $(f(\alpha), g(\alpha)) = (f(\beta), g(\beta))$, then we have a closed curve. If the functions defining the curve are *differentiable*, the curve is called smooth. Accordingly, if they are piece-wise differentiable then the curve is called piece-wise smooth. A plane curve

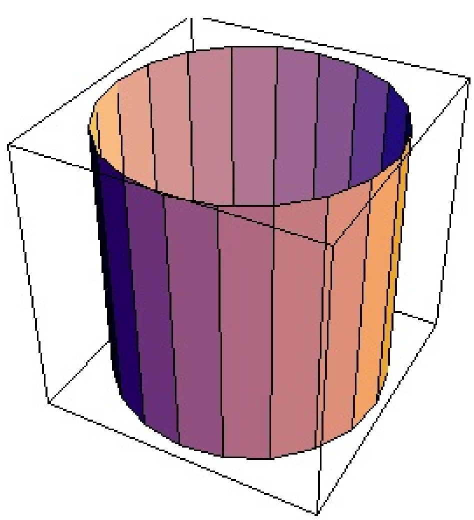
could be also given by a *polar* equation. See also *boundary curve*.
(2) A curve in R^3 is a set of all ordered triples $(f(t), g(t), h(t))$ with all functions continuous on some interval I. The definitions of closed, simple, and smooth curves remain the same as above. For calculations of lengths of curves see *length of parametric curve, length of polar curve, length of a space curve*.



cycloid A plane curve that appears as a trace of a point on a circle, when the circle rolls on a straight line. The parametric equations for the cycloid are given by

$$x = a(\theta - \sin\theta), \;\; y = a(1 - \cos\theta),$$

where a is the radius of the circle and θ is the angle formed by two radii in the circle, one connecting the center with the tracing point on the cycloid and the other connecting the center with the point where the circle touches line.



cylinder Geometrically, a cylinder is a three dimensional surface that consists of lines that are parallel to some given line and pass through some closed *plane curve*. The most commonly used cylinders are right circular cylinders that pass through a circle on the plane and are perpendicular to that plane. Algebraically, a right circular cylinder is given by the formula $x^2 + y^2 = r^2$, where the third variable z is free to assume any real values. Another example of a cylinder is a right elliptic cylinder given by the formula of *ellipse* $x^2/a^2 + y^2/b^2 = 1$ and the variable z is again free.

cylindrical coordinates A *coordinate system* in three dimensional space that uses polar coordinates in two dimensions and the *rectangular coordinate* in the third dimension. The relations are given by the formulas

$$x = r\cos\theta, \; y = r\sin\theta, \; z = z.$$

Here $r^2 = x^2 + y^2$ and the angle is determined from the equation $\tan\theta = y/x$.

cylindrical shell A geometric solid which is the difference of two concentric circular cylinders with the same axis but different radii. For applications of cylindrical shells see *volume*.

D

damped vibration If an object oscillates and another force (such as friction) affects its motion by decreasing its *amplitude*, then this results in a damped vibration of the object. This kind of motion is described by the differential equation

$$my'' + cy' + ky = 0,$$

where c is the damping constant.

data A set of values collected in the process of *sampling* or a *census*. The values of the data could be numerical (*quantitative*) or *categorical (qualitative)*.

data analysis A collection of statistical methods designed to evaluate and draw conclusions from a body of data. For specific methods see, e.g. confidence interval, hypothesis testing, analysis of variance, etc.

decimal numbers The numeric system with base ten. This system uses ten digits 0,1,2,3,4,5,6,7,8,9 and the place value system, which takes into consideration the position of the digit along with its value, to write all real numbers. For example, in the number 3531 the two 3's have different values because of being in different positions. The first 3 represents 3000 and the second one 30. To represent fractional and even irrational numbers in decimal system, we use the decimal point (in many countries a coma is used), which separates the whole part of the number from the fractional. In the number 43.587 the 43 is the whole part and the digits after the point represent the fractional part. Hence, $43.587 = 43 + 0.587$. Any *fraction (rational number)*, could be written as either a *terminating* or *non-terminating, repeating* decimal. On the other hand, any irrational number can be written as an *infinite non-repeating decimal*. Examples: $3/8 = 0.375$, $1/3 = 0.333...$, $\sqrt{2} = 1.414213562....$ See also *binary numbers*.

deciphering matrix The matrix used to decipher a coded message, usually, the inverse of the matrix that coded the message.

decomposition of matrices Writing a given matrix as a sum or product of two or more matrices. For a specific way of splitting a matrix see LU(lower upper)-decomposition.

decreasing function The function $f(x)$ defined on some interval I is decreasing, if for any two points $x_1, x_2 \in I$, $x_1 < x_2$, we have $f(x_1) > f(x_2)$. The function $f(x) = 2^{-x}$ is an example of a decreasing function.

decreasing sequence A sequence of real numbers $\{a_n\}$, $n \geq 1$, is decreasing, if $a_m < a_k$ whenever $m > k$. The sequence $a_n = 1/n^2$ is an example of a decreasing sequence.

definite integral (1) Let the function $f(x)$ be defined and continuous on some finite interval $[a, b]$. Definite integral of a function could be defined in many different but equivalent ways and we present some of this definitions.
Let us divide the interval into n equal parts of size $\Delta x = (b - a)/n$. Denote the endpoints of these parts by $x_0(= a), x_1, x_2, \cdots, x_n(= b)$ and chose arbitrary points $x_1^*, x_2^*, \cdots, x_n^*$ in each of these smaller intervals $[x_{i-1}, x_i]$, $1 \leq i \leq n$. Then the definite integral of f on the interval $[a, b]$ is the limit

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

if the limit exists. In special cases of this definition the sample points x_i^* can be chosen to be the left endpoints or right endpoints, or the midpoints of intervals. They all are equivalent to the more general definition.
The sum in the definition of the integral is called a *Riemann sum* and the definite integral is called the *Riemann integral*. The calculations for definite integrals are done primarily by the use of the Fundamental Theorem of Calculus, rather than using the definition directly. Examples:

$$\int_1^4 4x^3 dx = x^4|_1^4 = 256 - 1 = 255,$$

$$\int_0^\pi \sin x dx = -\cos x|_0^\pi = -(-1) - (-1) = 2.$$

For a positive function $f(x) \geq 0$ on $[a, b]$ the definite integral is just the area under the graph of this function. More generally, the definite integral is the difference of the areas above and below the x-axis for the graph of a given function.
(2) This definition could be used to include also functions with finite number of jump discontinuities. The notion of the definite (Riemann) integral could be generalized to include more general functions, such as with infinite discontinuities or defined on *infinite intervals*. See improper integral.
(3) Further generalizations of the definite integral resulted in development of other types of integrals (Stiljes, Lebesgue, Denjoy, etc.).

definite integration The process of calculating the *definite integral* of some function.

degenerate conic sections Special cases of *conic sections*, when the plane cutting the cone produces a point or a pair of intersecting lines. Algebraically, these cases correspond respectively to the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0, \quad y^2 = a, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

degree of freedom If n unknowns (or variables) $x_1, x_2, \cdots, x_n$ are connected by one relation, then only $n-1$ of them can be chosen arbitrarily, whereas the nth one is dependent on the other choices. For that reason we say that the degree of freedom of these variables is $n-1$. If the same unknowns are connected by two relations, then the degree of freedom would be $n-2$, and so on.

degree of a polynomial For a general polynomial given by the formula

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

the whole number $n \geq 0$. Example: For the polynomial $p(x) = 3x^5 - 2x^4 + 5x^2 - 6$ the degree is 5.

degree measure of an angle One of the two main units for measuring an angle (the other unit is called radian measure). In this system the circumference of any circle is divided into 360 equal *arcs* and the angle subtended on one of those arcs is said to have the measure 1°. Hence, the right angle would have 90° and the straight angle 180°. See also *angle*.

demand function Also called price function. The price of the product when a company wants to sell x units of that product, usually denoted by $p(x)$. The graph of a demand function is called a demand curve. Closely related are *revenue function, profit function*.

DeMoivre's theorem Let the complex number z be written in trigonometric form: $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$. Then for any integer n $z^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$.

denominator (1) For the numeric fraction $\frac{a}{b}$, the number b. (2) For the rational function $\frac{p(x)}{q(x)}$, the polynomial q. (3) For any expression of the form $\frac{f(x)}{g(x}$ the function $g(x)$.

dense set A subset of real numbers is dense, if for any point a of that set there are infinitely many other points b of the same set, arbitrarily close to a. Formally, the set S in R is dense, if for any point $a \in S$ and any number $\varepsilon > 0$, there exists another point $b \in S$, such that $|a - b| < \varepsilon$. Examples: The set of all rational numbers is dense, but the set of all integers is not.

density function Also called probability density function. A function that represents some type of distribution. The graph of a density function is called a density curve. Density function must be non-negative and that the area under a density curve is exactly 1, since it represents the total probability. Symbolically, a function $f(x)$ is a density function if $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x)dx = 1.$$

dependent variable In an equation $y = f(x)$ the variable y. The variable x is called the *independent variable*.

derivative Let the function $f(x)$ be defined on some interval (a, b). The derivative of the function at some point $x = c$ is defined as the limit

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

if this limit exists. The derivative of a function is a function itself given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

where x is any point in the domain of the function such that this limit exists. The set of all these points is the domain of the derivative. Some other common notations for the derivative of $y = f(x)$ are:

$$y',\ \frac{dy}{dx},\ \frac{df}{dx},\ D_x f,\ Df(x).$$

In practice, derivatives are calculated by the differentiation rules, rather than using the definition directly. Examples:

$$(x^n)' = nx^{n-1}, \quad (\tan^{-1} x)' = \frac{1}{1+x^2}, \quad (e^x)' = e^x.$$

See also *left-hand derivative, right-hand derivative.* For derivatives of functions of several variables see partial derivatives, *directional derivatives, normal derivatives.*

derivative of a composite function See chain rule.

derivative of indefinite integral Let f be a continuous function on some interval $[a, b]$ and $F(x) = \int_a^x f(t)dt$. Then $F'(x) = f(x)$ for $a < x < b$. This result is one case of the *Fundamental Theorem of Calculus.* This statement has far reaching generalizations for functions $f(x)$ other than continuous functions.

derivative of the inverse function Let $f(x)$ be a differentiable function on some interval I and let $g(x)$ be its *inverse.* Then, if for some point $a \in I$, $f'(g(a)) \neq 0$, then the derivative of g at that point is given by

$$g'(a) = \frac{1}{f'(g(a))}.$$

This formula makes it possible to calculate derivatives of functions if we know the derivatives of their inverse. Example:

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}},$$

because we know that $(\sin x)' = \cos x$ and $\cos(\sin^{-1} x) = \sqrt{1-x^2}$.

derivative of a vector function See *differentiation of vector functions.*

Descartes' rule of signs Let the polynomial of degree $n \geq 1$ be written in the standard form:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

and $a_0 \neq 0$. Then (1) The number of positive zeros of p is either equal to the number of sign changes of coefficients of $p(x)$ or is less than that number by an even integer; (2)The number of negative zeros of p is either equal to the number of sign changes of coefficients of $p(-x)$ or is less than that number by an even integer. Example: The polynomial $p(x) = 2x^3 - 5x^2 + 6x - 4$ has three sign changes, so it may only have three or one positive zeros (roots). Next, $p(-x) = -2x^3 - 5x^2 - 6x - 4$, has no sign changes, so p cannot have any negative zeros.

determinant A number, associated with any *square matrix.* For a given matrix A the notations for corresponding determinant are $|A|$ or $\det(A)$. The value of the determinant is defined recursively as follows: If the matrix is 1×1, $A = (a)$, then $|A| = a$. For the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$|A| = ad - cb$. In general, for the $n \times n$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

the determinant is equal to

$$|A| = \sum_{j=1}^{n} a_{1j} C_{1,j},$$

where $C_{1,j}$ are the first row cofactors. This presentation is called expansion by the first row. It is possible to expand the determinant by any row or column and the formulas are similar. Example:

$$\begin{vmatrix} -2 & 2 & 3 \\ 1 & -3 & -4 \\ -4 & 0 & 1 \end{vmatrix} = (-2)\begin{vmatrix} -3 & -4 \\ 0 & 1 \end{vmatrix}$$

$$-2\begin{vmatrix} 1 & -4 \\ -4 & 1 \end{vmatrix} + 3\begin{vmatrix} 1 & -3 \\ -4 & 0 \end{vmatrix}$$

$$= (-2)\cdot(-3) - 2\cdot 17 + 3\cdot 12 = 8.$$

Equivalent definition of determinants is possible with the use of permutations and *signed elementary products*. With this approach, the determinant of the matrix is defined as the sum of all possible signed elementary products and the signs depend on the fact of the permutation being odd or even.

determinant function The same as determinant of a *square matrix*. Equivalent definition: Determinant function is the sum of all *signed elementary products*.

deviation In statistics, the difference between an individual value of a set and the *mean* of all set of values under consideration. See also standard deviation and variance.

diagonal In geometry, in case of *polygons* or *polyhedra*, diagonal is understood as any segment, connecting the vertices of that objects.

diagonal matrix A matrix, that has its only nonzero elements on the main diagonal. The general diagonal matrix has the form

$$\begin{pmatrix} a_1 & 0 & 0 & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ 0 & 0 & a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_n \end{pmatrix}$$

diagonalization of matrices A square matrix A is diagonalizable, if there exists an invertible matrix P such that $P^{-1}AP = D$ and D is a *diagonal matrix*. The main theorem in the problem of diagonalization states that a matrix is diagonalizable if and only if it has a system of *linearly independent eigenvalues*. Additionally, the matrix P could be chosen in such a way that that eigenvalues be the elements on the diagonal of D. The method of diagonalization is also used in solving homogeneous or non-homogeneous differential equations.

diameter For a *circle*, the line segment passing through the center an connecting two opposite points on the *circumference*. Standard formulas involving diameter are: $d = 2r$, r is the *radius* of the circle, $C = \pi d$, C is the length of circumference.

difference When performing the operations of subtracting one number from the other, one function from the other, one matrix from the other, etc., the result is the difference of given objects.

difference quotient For a given function $f(x)$ the expressions

$$\frac{f(x) - f(a)}{x - a} \quad \text{or} \quad \frac{f(x+h) - f(x)}{h},$$

where $x \neq a$ and $h \neq 0$. These expressions are the important part of definition of the *derivative* of the function.

difference law of limits If the functions $f(x)$ and $g(x)$ both have finite limits when x approaches some point a, then the limit of the difference function is also finite and

$$\lim_{x\to a}(f - g)(x) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x).$$

See also limit laws.

difference rule For *differentiation*. If $f(x)$ and $g(x)$ are differentiable, then

$$[(f - g)(x)]' = f'(x) - g'(x).$$

See also differentiation rules.

differentiable function at a point $x = c$. A function, that is defined at a neighborhood of the point c and for which the first *derivative* exists. If that property is true for all points of some interval (a, b), then the function is called differentiable on that interval.

differential (1) For functions of one variable. Let $y = f(x)$ be a *differentiable function*. Then

$$df = dy = f'(x)dx$$

is called the differential.
(2) For functions of several variables. Let $f(x,y,z)$ be differentiable. Then

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

is the differential of f. Differential of a function of two or n variables is defined similarly. In multidimensional case the differential is also called *total derivative.*

differential equation An equation that involves the unknown function along with its derivatives of some order. These equations are classified depending on order of derivative, the type of coefficients and many other characteristics. The equation

$$2x^2y'' - \sin xy' + \ln y = 3x$$

is a non-homogeneous second order linear ordinary differential equation with variable coefficients. In cases when the right side is zero (i.e., only the functions $y, y'y''$, etc. are involved), the equation is called homogeneous. If any of the "variables" $y, y'y'', \cdots$ are present in a non-linear form, then the equation is called non-linear. When functions of several variables and their partial derivatives are involved, equation is called partial differential equation. The solution of differential equation is any function, that substituted into the equation makes it an identity. The general solution of an equation is a solution that contains all possible solutions.

differential operator An operation that assigns to a given function an expression containing that function and some of its derivatives. The expression

$$L[\phi](x) = \phi'' + p(x)\phi' + q(x)\phi,$$

where $\phi(x)$ is twice differentiable function is an example of second order differential operator. The order of the operator is determined by the highest derivative involved.

differentiation The process of calculating the *derivative* of some function. See also *implicit differentiation, partial differentiation.*

differentiation rules Also called differentiation formulas. These are the rules that allow to calculate *derivatives* of functions easily, without relying on its definition. Here are the basic rules.
Let $f(x)$ and $g(x)$ be two *differentiable functions* on some interval. Then for derivatives of their sum, difference, product, and quotient we have:

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}.$$

For derivative of *composite functions* see *chain rule* and for derivatives of *implicit functions* see *implicit differentiation.*

differentiation of a vector function Let $\mathbf{r}(t)$ be a three dimensional vector function. The derivative of this function is defined very similar to the derivative of scalar functions:

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if the limit exists. Vector differentiation follows the same rules as for scalar functions in the sense that addition, subtraction, scalar multiplication rules are identical. However, since there is no notion of division of vectors, we do not have quotient rule. Furthermore, multiplication has two versions for three dimensional vectors: scalar product and cross product. The product rules in both cases are similar: If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are two vector functions, then

$$\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t),$$

$$\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t).$$

All these rules are also valid for vector functions in any dimensional space except the last one because the cross product is defined in R^3 only. Additionally, the chain rule has the following form:

$$\frac{d}{dt}[\mathbf{u}(f(t)) = f'(t)\mathbf{u}'(f(t)).$$

digits In *decimal* numeric system the numerals $0, 1, 2, 3, 4, 5, 6, 7, 8$ and 9.

dilation operator Or dilation transformation. An operator that applied to a vector does not change its direction but makes its norm bigger. The operator is given by the formula $T\mathbf{x} = k\mathbf{x}$, where $k \geq 1$. In the case $0 \leq k \leq 1$ the transformation is called *contraction operator*. For a more general case when not only the norm but also the direction could be changed, see *expansion operator*.

dimension of vector space If a vector space V has a system of linearly independent vectors $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$, that span V, then the number n is the dimension of the vector space. Also, any basis of V, has the same number of vectors. In the case, when there is no finite basis, the space is called infinite dimensional.

dimension theorem for linear transformations Similar to *dimension theorem for matrices.* If T is a linear transformation from the n-dimensional vector space V to some other vector space W, then

$$rank(T) + nullity(T) = n.$$

dimension theorem for matrices If A is a matrix with n columns, then

$$rank(A) + nullity(A) = n.$$

Here $rank(A)$ is the rank of the matrix and $nullity(A)$ is the dimension of the kernel of the matrix.

Dirac delta function A generalized "function", formally defined for any real point c by the conditions:

$$\delta(t - c) = 0,\ t \neq c, \quad \int_{-\infty}^{\infty} \delta(t - c)dt = 1.$$

The *Laplace transform* of this function could be calculated in generalized sense and is equal to e^{-sc}. Additionally, *convolution* of any function with Dirac function results in the value at point c:

$$\int_{-\infty}^{\infty} \delta(t - c)f(t)dt = f(c).$$

This function plays very important role in theoretical phisics.

direct variation The name of many possible relations between two variables. The most common are:(1) y varies directly with x means that there is a real number $k \neq 0$, such that $y = kx$; (2) y varies directly with x^2 means $y = kx^2$; (3) y varies directly with x^3 means $y = kx^3$. There are many other possibilities but rarely used. Compare also with *inverse variation* and *joint variation.*

directed line segment The same as vector.

directional derivative For functions of two real variables. Let $\mathbf{u} = (a, b)$ be a *unit vector* in the plane and $f(x, y)$ be a function. The derivative of f in direction of the vector $\mathbf{u}$ at some point (x_0, y_0) (denoted by $D_{\mathbf{u}}f(x_0, y_0)$) is the limit

$$\lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h},$$

if it exists. The definition for functions of three or more variables is similar.

direction angles The angles any nonzero vector $\mathbf{v}$ makes with the *coordinate axes* are called direction angles. The cosines of these angles are the direction cosines. Hence, if α is the angle between $\mathbf{v}$ and x-axis, then

$$\cos \alpha = \frac{\mathbf{v} \cdot i}{|\mathbf{v}||i|}$$

and similarly for other two directions. If a vector is used to describe the direction of some line then the direction cosines are also called direction numbers.

direction field For differential equations such as the equation $y' = f(t, y)$. We can make a direction field by choosing several points on the *coordinate plane* and by drawing a small line segment from each point with the slope equal to the value of f at that point. This method is a useful tool when trying to get an idea about the behavior of the solution which is difficult (even impossible) to find.

directrix In geometric definition of the *parabola*, the name of the fixed line with the following property: A parabola is the set of all points on a plane that are equidistant from a fixed point (called *focus*) and a fixed line, not containing that point is called directrix. In a more general view, directrix could be defined also for ellipses and hyperbolas. See eccentricity.

discontinuity Property for a function, opposite to being continuous. A function could be discontinuous by different reasons. For example, the function

$$f(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$$

is discontinuous at the point $x = 0$, because the *left-hand* and *right-hand limits* at that point are not equal. The function $f(x) = 1/x$ is discontinuous at the point $x = 0$, because it is not defined there. See also *discontinuous function*.

discontinuous coefficients Usually applies to differential equations with variable coefficients. In many cases these kind of equations can be solved (or shown to have a solution) even if the coefficients have some type of discontinuity, usually *jump discontinuity*.

discontinuous function A function $f(x)$ is discontinuous at some point $x = c$, if one of the following happens. (1) The limit $\lim_{x\to c} f(x)$ does not exist. (2) The limit exists, but does not equal the value of the function at that point: $\lim_{x\to c} f(x) \neq f(c)$. (3) The function is defined around that point, but not at the point.
This property is opposite to the property of being *continuous*.

discrete variable A variable that does not take its values continuously. Being discrete could be expressed in different ways. For example, if the variable x takes whole values $0, 1, 2, 3, \cdots$ or it takes *rational values*, then it will be discrete in both situations. As a rule, however, we usually consider random variables taking either whole or integer values.

discriminant For the general quadratic function

$$f(x) = ax^2 + bx + c$$

the quantity $D = \sqrt{b^2 - 4ac}$ is the discriminant. Depending on the sign of D, the corresponding quadratic equation $f(x) = 0$ has: (1) Two distinct real roots, if $D > 0$; (2) Two real repeated roots, if $D = 0$,or (3) Two complex conjugate roots, if $D < 0$.

disjoint events Two events that cannot happen at the same time. In symbolic form, if A and B are the events, then they are disjoint if $A \cap B = \emptyset$.

disk The term is used as a synonym for *circle*. The spelling disc is also valid.

displacement Some kind of shift or move from the given position.

distance One of the most important geometric (and also physical) notions. The distance is understood as a measure of how far or close two objects are. Different measuring methods and also different measuring units, such as *metric* or *English* measuring units, may be used. In Algebra and Calculus we measure distances by the *distance formula* or by many other methods, such as arc length formula, involving integration. See corresponding entries for details.

distance between a point and a line If (a, b) is some point on the plane and the equation $Ax + By + C = 0$ represents a line, then the distance of the point from the line is given by the formula

$$d = \frac{|Aa + Bb + C|}{\sqrt{A^2 + B^2}}.$$

distance between a point and a plane Let the equation of a plane be given by $Ax+By+Cz+D = 0$ and the point $P(x_0, y_0, z_0)$ is not on the plane. The distance now will be given by the formula

$$d = \frac{Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

distance formula Let (x_1, y_1) and (x_2, y_2) be two points on the plane. Then the distance between them is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Similarly, if $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ are two vectors in *Euclidean* space R^n, then the distance is given by a similar formula

$$d = \sqrt{\sum_{k=1}^{n} (y_k - x_k)^2}.$$

distance problem The problem of finding the distance traveled by a moving object. If the object moves with a constant speed v, then the distance, s is given by $s = v \cdot t$, where t is the time of travel. In general, if the velosity of the object is variable and is given by the function $f(t)$, then the distance traveled between time periods $t = a$ and $t = b$ will be given by the integral

$$s = \int_a^b f(t)\, dt.$$

distribution More precisely, probability distribution. For any random event, the outcomes appear with certain frequencies. The combination of all outcomes with their relative frequences is a distribution. Depending on types and number of outcomes, distributions are divided into discrete and continuous. Any discrete distribution has only finite number of outcomes and continuous distributions have infinitely many, but not discrete outcomes.
(1) Discrete distributions. Every outcome has a probability (frequency) of appearing, that is either positive or zero. The sum of probabilities of all possible outcomes is 1. Formally, if there are n outcomes $X_1, X_2, \cdots, X_n$ and $P(A)$ indicates the probability of an event A, then $P(X_i) \geq 0,\ i = 1, 2, \cdots, n$ and $\sum P(X_i) = 1$. Binomial distribution is one of the most common discrete probability distributions and histograms are their most common visual representations.
(2) Continuous distributions are represented by a density curve which is the graph of a positive function with the property that the area under that curve is 1. Individual probability of any outcome is always zero (because there are infinitely many outcomes), hence the probabilities in question are probabilities of groups of events. This means that if X is any random outcome, then $P(X) = 0$ but, for example $P(a \leq X \leq b) \geq 0$. Additionally, as in the case of discrete distributions, the probability of all space of outcomes is 1. For specific examples of continuous distributions see *normal distribution, Poisson distribution, t-distribution, F-distribution, uniform distribution.*

distributive property For any real or complex numbers $a,\ b,\ c$, the property

$$a(b + c) = ab + ac,$$

that allows to group, or to split algebraic or numeric expressions. Similar properties are valid also for other mathematical objects, such as functions, matrices, etc.

divergence of a vector field For a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ in R^3 the divergence is a *scalar* function, defined by the equation

$$div\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

if the partial derivatives exist. See also curl of a vector field and gradient vector field.

divergence test for series Let $\{a_n\}$ be a sequence of numbers. If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the *series* $\sum_{n=1}^{\infty} a_n$ is divergent.

divergence theorem Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a *vector field* with continuously differentiable components in some open region of R^3. Let D be a region,contained in the domain of $\mathbf{F}$, with the boundary S. Then, if $\mathbf{n}$ is the outward normal to S, we have the relationship

$$\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int\int\int_D div\mathbf{F}\, dV.$$

Also called *Gauss' theorem.*

divergent improper integral (1) The improper integral $\int_a^\infty f(x)dx$ is divergent, if the limit

$$\lim_{A\to\infty} \int_a^A f(x)dx$$

does not exist or is infinite. The definitions for integrals $\int_{-\infty}^b f(x)dx$ and $\int_{-\infty}^\infty f(x)dx$ are similar.

(2) For improper integrals of unbounded functions on bounded intervals $\int_a^b f(x)dx$, where f has infinite limit at the point $x = b$. We call that integral divergent, if the limit

$$\lim_{\delta \to 0} \int_a^{b-\delta} f(x)dx$$

does not exist or is infinite. The definitions for the cases when the point of unboundedness of the function is the left endpoint or some inside point, is similar.

divergent power series Each power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n$$

has an interval of convergence given by $|x - a| < R$, where $R \geq 0$. For any values of x, $|x - a| > R$, the series will be divergent. In case when $R = 0$, the series will be convergent in one point only, $x = a$. See also *divergent series* and *convergent series*.

divergent sequence A sequence, where the terms do not go to any particular finite limit. The opposite of the *convergent sequence*. Examples: The sequences $1, -1, 1, -1 \cdots$ and $1, 4, 9, ..., n^2, ...$ are divergent.

divergent series A numeric series $\sum_{n=1}^{\infty} a_n$ is divergent, if the sequence of its *partial sums* $S_n = \sum_{i=1}^{n} a_i$ does not have a finite limit, i.e. is a *divergent sequence*. The opposite of *convergent series*. Examples: The *harmonic series* is divergent, because its partial sums grow indefinitely large. The series $1 - 1 + 1 - 1 + 1 - \cdots$ is divergent, because its partial sums do not approach any particular limit.
The notion of divergence applies also to functional series such as *power series* or *Fourier series*. A functional series is divergent at some point if the numeric series formed by the values of functional series at that particular point is divergent.

dividend In the division process of two numbers, the number, that is getting divided. The other number is called *divisor*.

divisible polynomial A *polynomial*, that can be divided by another polynomial. By the *Fundamental Theorem of Algebra*, any polynomial of *degree* two or higher is divisible by another polynomial, with possibly complex coefficients. If we restrict division to be by real polynomials only, then not all the polynomials will be divisible. Examples: The polynomial $p(x) = x^2 - 5x + 6$ is divisible by real polynomials $x - 2$ and $x - 3$. The polynomial $q(x) = x^2 + 4$ is not divisible by any real polynomial but it is divisible by complex polynomials $x + 2i$ and $x - 2i$.

division One of the four basic operations in algebra and arithmetic. By dividing two numbers we get their *ratio*. Division is possible by any real (or complex) number, except zero. Division of any number by zero is not defined.

division of complex numbers (1) To divide two complex numbers $\frac{a+ib}{c+id}$, written in the standard form, we multiply both numerator and denominator by the *conjugate* of the denominator, $c - id$. This allows to get a *real number* $c^2 + d^2$ in the denominator and divide the resulting complex number in the numerator by that real number:

$$\frac{a+ib}{c+id} = \frac{ac-bd}{c^2+d^2} + \frac{ad+bc}{c^2+d^2}i.$$

(2) If the complex numbers $z = r(\cos\theta + i\sin\theta)$ and $w = \rho(\cos\phi + \sin\phi)$ are given in trigonometric form, then

$$\frac{z}{w} = \frac{r}{\rho}[\cos(\theta - \phi) + i\sin(\theta - \phi)].$$

division of fractions To divide the *fraction* a/b by another fraction c/d, we just multiply the first one by the *reciprocal* of the second:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

See also *multiplication of fractions*.

division of functions For two functions $f(x)$ and $g(x)$ their quotient (division function) is defined to be the quotient of their values: $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$. This function is defined where f and g are defined, except where $g(x) = 0$.

division of polynomials Let $P(x)$ and $Q(x)$ be two polynomials and assume that degree of Q is less or equal to the degree of P. To divide P by Q, means to find out "how many times Q fits into P". As in the case of division of integers, the result is not always a specific polynomial, but rather there is some remainder. By the division algorithm, in this case we have

$$\frac{P(x)}{Q(x)} = D(x) + \frac{R(x)}{Q(x)},$$

where $R(x)$ is the remainder of this division. Example:

$$\frac{5x^3 - 4x^2 + 7x - 2}{x^2 + 1} = 5x - 4 + \frac{2x + 2}{x^2 + 1}.$$

In practice, division process is done either by long division , or by synthetic division algorithms. The second one is possible only if we need to divide by a binomial of the form $x - c$. See corresponding entries for the details of the algorithms.

divisor In the division process of two numbers, the number, that divides the other one, called *dividend*.

domain of a function For a function of one real variable $f(x)$, the set of all real values x, such that the function is defined and is finite. More generally, for a function $f(x_1, x_2, \cdots, x_n)$ of n independent variables, the region G in the Euclidean space R^n, such that f is defined for all values of variables that belong to the region G.
Examples: For the function of one variable

$$f(x) = \frac{1}{\sqrt{x - 1}}$$

the domain is the set $\{x|x > 1\}$ because the square root function is defined for non-negative values of argument and, additionally, division by zero is not defined.
For the function of two variables

$$f(x, y) = \ln(x^2 + y^2)$$

the domain is all values of x and y except $(0, 0)$ because the logarithmic function is not defines at the origin.

dot product For any two vectors $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)$ in *Euclidean* space R^n, the number

$$\mathbf{x} \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Dot product is a special case of *inner product*. See also *scalar product*.

double angle formulas For trigonometric functions the formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x =$$

$$2 \cos^2 x - 1 = 1 - 2 \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

Similar formulas are valid for other trigonometric functions but are rarely used.

double integral (1) Let a function $f(x, y)$ of two variables be defined on some rectangular region $D = [a, b] \times [c, d] = \{(x, y)|a \le x \le b,\ c \le y \le d\}$. Then the double integral of the function in the domain D is the limit of its *Riemann sums*, as the number of partitions of the sides of rectangle goes to infinity:

$$\int\int_D f(x, y) dA = \int\int_D f(x, y) dx dy$$

$$= \lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*) \Delta x \Delta y.$$

(2) In the case the function is given on a more general region than a rectangle, the definition requires some modifications. Assume that the region has the form

$$D = \{(x, y)|a \le x \le b, g_1(x) \le y \le g_2(x)\},$$

where $g_1,\ g_2$ are continuous on $[a, b]$. Then the double integral on this type of regions is defined to be the *iterated integral*

$$\int\int_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

(3) Integrals over the regions of the type

$$D = \{(x, y)|c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

are defined similarly. See also definite integral, *triple integral, multiple integral.*

double Riemann sum Let a function $f(x, y)$ of two variables be defined on some rectangular region $D = [a, b] \times [c, d] = \{(x, y)|a \leq x \leq b,\ c \leq y \leq d\}$. In analogy of forming *Riemann sums*, we divide interval $[a, b]$ into n equal parts of size $\Delta x = (b-a)/n$ and interval $[c, d]$ into m equal parts of size $\Delta y = (d-c)/m$ by choosing points $x_0(= a), x_1, \cdots, x_n(= b)$ and $y_0(= c), y_1, \cdots, y_m(= d)$. In resulting smaller rectangles $D_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j],\ 1 \leq i \leq n,\ 1 \leq j \leq m$, we choose some points (x_i^*, y_j^*) and form the double sum

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta x \Delta y,$$

which is called double Riemann sum. The choice of points (x_i^*, y_j^*) is arbitrary, and (under some restrictions on function) always results in the same limit. See also double integral.

E

e One of the most important universal constants in mathematics. This is a non-algebraic, transcendental number, such that there are many approximation formulas. Among them, most familiar are:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

and

$$e = \lim_{n\to\infty} (1 + \frac{1}{n})^n.$$

The decimal approximation of e is 2.7182818...

eccentricity For conic sections. Let L be a fixed line on the plane (called *directrix*) and P be some point (called *focus*), outside of that line. A positive real number e is now fixed. The set of all points Q on the plane, satisfying the relation

$$\frac{dist(Q, P)}{dist(Q, L)} = e$$

is called conic section and e is its eccentricity. In case $e < 1$ the result is *ellipse*, for $e = 1$ we get *parabola* and when $e > 1$, the result is *hyperbola.* For ellipses and hyperbolas the eccentricity has a simpler description. See respective definitions.

echelon form of a matrix Also called row echelon form. The result of performing Gaussian elimination process on the matrix. In echelon form each row should start with zero or the *leading one*. Further, in each next row the leading one should be to the right of the previous row's one. Finally, the rows with all zero elements should appear in the bottom rows only. Example: The matrix

$$\begin{pmatrix} 1 & 4 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is in row echelon form. The matrix is said to be in reduced row echelon form, if, additionally, the only non-zero entries are the leading ones.

eigenspace Let λ be an *eigenvalue* of a given square matrix A and assume $\{\mathbf{x_1}, \cdots, \mathbf{x_k}\}$ are all linearly independent *eigenvectors* of that matrix corresponding to the same eigenvalue. Linear space spanned by that vectors is called the eigenspace for the eigenvalue λ.

eigenvalues of matrix For a square matrix A, a real or complex number λ, such that there exists a non-zero vector $\mathbf{x}$ satisfying the equation $A\mathbf{x} = \lambda\mathbf{x}$. Eigenvalues are roots of the characteristic equation. An eigenvalue is called simple if it appears only once in the factorization of the characteristic polynomial. There are two different notions of multiplicity for eigenvalues. The number of times $\lambda - \lambda_0$ appears in the factorization of the characteristic polynomial is its algebraic multiplicity. On the other hand, the dimension of the *eigenspace*, corresponding to the value λ_0 is its geometric multiplicity. According to a theorem, the geometric multiplicity is always less than or equal to its algebraic multiplicity. See also *eigenvectors of matrix* below.

eigenvectors of matrix A non-zero vector $\mathbf{x}$, satisfying the equation $A\mathbf{x} = \lambda\mathbf{x}$ for some square matrix A and real or complex number λ. The number λ is called an *eigenvalue*. Eigenvectors, corresponding to different eigenvalues are orthogonal (and hence, linearly independent). See also *eigenvalues of matrix*.

element of a matrix The numbers (or functions) that form a *matrix*. See also *entries of a matrix*.

element of a set Members of the given set. Depending on nature of the set the elements can be numbers, functions, matrices or any other mathematical objects.

elementary matrices Any matrix, which could be received from the *identity matrix* by one *elementary row operation*. Example: The matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

is elementary, because it is the result of adding the first row to the third row of the identity matrix.

elementary product For a *square matrix* A of size n, elementary product is any product of n elements from that matrix that contains exactly one element from each row and each column. See also *signed elementary product*.

elementary row operations For systems of algebraic linear equations or matrices the three elementary row operations are:
(1) Interchanging any two rows;
(2) Multiplying any row by any non-zero constant;
(3) Adding non-zero multiple of any row to any other row.
These operations are performed as a part of *elimination method* of solving systems of linear equations.

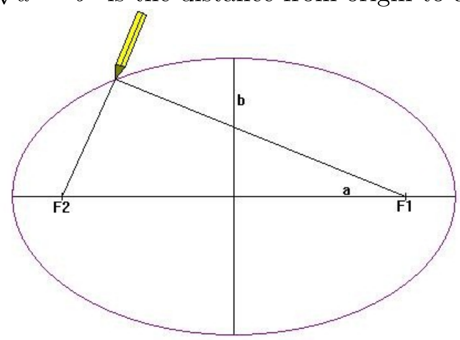
elimination method One of the methods of solving systems of equations with more than one variable. The method constitutes of eliminating one of the variables in one or more equations, thus arriving to equation (or system of equations) with less variables. This approach is mainly used for solving systems of linear equations but can also be used for solving non-linear systems. For specific description of this method for solving systems of linear equations, see Gaussian elimination or Gauss-Jordan elimination.

ellipse One of the three main *conic sections*. Geometrically, an ellipse is the location (locus) of all points in a plane, the sum of whose distances from two fixed points in the plane is a positive constant. The two points, from which we measure distances, are called *foci* (plural for *focus*). Equivalently, the ellipse could be described as the result of cutting *double cone* by a plane not parallel to its axis or any of the *generators of the cone*, and not passing through vertex of the cone. In the special case, when the plane is perpendicular to the axis, the result is a *circle*. An ellipse is a bounded *smooth curve*. Alternative geometric definition could be given with the use of *eccentricity*. See corresponding definition.
Algebraically, the general equation of an ellipse is given by the quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, C, D, E, F are real constants and $A \cdot C > 0$. In the case, when the foci are located on one of the coordinate axes and

center coincides with the origin, this equation could be transformed into *standard form*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where the constants $a,\ b$ have special meaning: $c = \sqrt{a^2 - b^2}$ is the distance from origin to a focus.



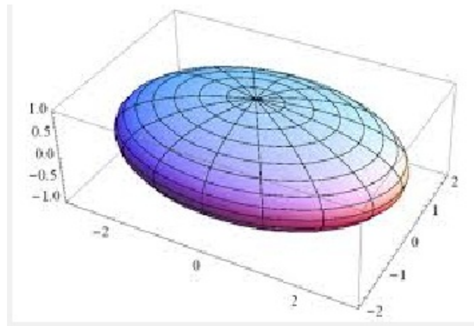
Moreover, the intersection points of ellipse with the coordinate axes are called *vertices* of ellipse and their distances from origin are equal to a and b. The line connecting two foci is called *major axis* and the perpendicular line (passing trough origin) is the *minor axis*.
In the more general case, when the center of ellipse is located at some point (h, k) but the major and minor axes are still parallel to coordinate axes, the standard equation becomes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

In these cases *eccentricity* is given by the formula $e = c/a$.
Both of the above cases happen when in the general equation the term Bxy is missing. In the case $B \neq 0$ the result is still an ellipse, which is the result of rotation of one of the previous simpler cases.
The ellipse could also be given by its polar equation:

$$r = \frac{de}{1 \pm e\cos\theta} \quad \text{or} \quad r = \frac{de}{1 \pm e\sin\theta},$$

where $d > 0$ and $0 < e < 1$ is the *eccentricity* of the ellipse.



ellipsoid The three dimensional surface given by the formula

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

elliptic cone The three dimensional surface given by the formula

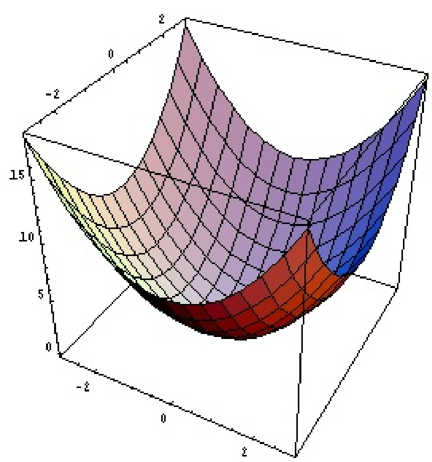
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

This surface represents a cone which perpendicular cuts are ellipses instead of circles. In particular case $a = b$ we get circular cone.

elliptic paraboloid The three dimensional surface given by the formula

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}.$$

The picture shows the paraboloid given by the equation $z = x^2 + y^2$.



empty set The set, that contains no elements. The notation is $\emptyset$.

endpoints For the closed or open intervals $[a, b]$, (a, b), the points a and b. The same is true for half-open intervals $[a, b)$ and $(a, b]$.

endpoint extreme value *Minimum* or *maximum* value of a function, that occurs at the *endpoint* of the *closed interval* $[a, b]$.

English system of measurements The system where the distances (also areas and volumes) are measured by inches(in), feet(ft), yards(y), and miles(mi) and the weights are measured using ounces(oz) and pounds(lb). Unlike the *metric system* of measurements which is based on decimal numeric system, relations in English system are more or less arbitrary. Here are some of that relations: 1 ft= 12in; 1 y=3 ft, 1 mi= 1760 y , 1 lb= 16 oz.

entries of matrix Same as *elements of a matrix.*

epicycloid When a circle rolls on the outside of a fixed circle, the path of any point is called epicycloid. Let a be the radius of the fixed circle with center at the origin and b be the radius of the moving circle. When we connect the origin (center of the first circle) with the center of the second circle and continue, the resulting radius in the moving circle will form some angle θ with the horizontal line. In this setting, the parametric equations of epicycloid are given by the formulas

$$x = (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta,$$

$$y = (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta.$$

equality The property of being equal to. Mathematically, equality means that certain quantity is equal another. These quantities may contain numbers, variables, functions, and other mathematical objects. If the equality holds for all values of given parameters, then we have the case of *identity.* If the equality is true for some values of that parameters only, then we have the case of *equation.* Equations are also called conditional equality. See corresponding entries.

equation A statement that one quantity equals another. Unlike *identities*, equations are not true for all the values of variables that are involved in that relationship and by that reason they are also called conditional equality. To solve an equation means to find all values of the variables or unknowns that make that statement an identity. Depending on the type of variables, the equations can be *polynomial, rational, trigonometric, radical, exponential, logarithmic*, and so on. In the case when the unknown quantity is a function, then we can have a *differential, integral* or *integro-differential* equation. There are also equations that contain *matrices, vectors* and other mathematical objects. The equations could be given both in *rectangular* coordinates and also in *polar, spherical* and other variables. Examples:
Polynomial: $3x^3 - 2x^2 + 4x - 5 = 0$;
Exponential: $2^{x-1} - 4 = 0$;
Trigonometric: $\sin x + \cos x = 1$;
Logarithmic: $\ln(x^2 - 1) + \ln(x^2 + 1) = 0$;
Polar: $r^2 + \sin\theta = 0$;
Differential: $2xy'' + 3x^2y' - y = 0$.
Some equations cannot be categorized because they contain different kinds of functions, such as the equation $2x^2 - 2^{x+1} + \sin x = 0$. These kind of equations are called mixed. See also corresponding entries for all the mentioned types of equations.

equilateral triangle A triangle that has all sides equal in size. In equilateral triangles all three angles are also equal in size and measure $60°$ ($\pi/3$ in radian measure).

equilibrium point or position In Physics or Mechanics the point or position where the system reaches equilibrum. This means that after reaching that point the system does no longer move unless some outside force is applied.

equilibrum solution For *autonomous* differential equations. When solving the equation $y' = f(y)$, some solutions turn out to be constants, hence, do not depend on time, and are called equilibrum. This happens exactly at the roots of equation $f(y) = 0$ and these roots are called *critical points.*

equivalence A notion that has many manifestations in mathematics. In the most general and ab-

stract form it could be defined as some kind of relation between elements of a set S that satisfies three *axioms* for any elements a, b, c of that set. If we denote that relation by "$\circ$", then the axioms are:
1. (Reflexivity) $a \circ a$
2. (Symmetry) if $a \circ b$ then $b \circ a$
3. (Transitivity) if $a \circ b$ and $b \circ c$ then $a \circ c$.
This way equality relation becomes equivalence, similarity of triangles is equivalence as well as others.

equivalent equations Two equations that have the same *solution sets*. Equations $x^2 - 5x - 6 = 0$ and $(x+1)(x-6) = 0$ are equivalent, because they have the same solutions $x = -1, 6$. The term is usually used for equations of the same type, but equations of different nature can be equivalent too. The equation $2x - 1 = 1$ is algebraic and $\log_2 x = 0$ is logarithmic, and they are equivalent because they have the same solution $x = 1$.

equivalent inequalities Two inequalities that have the same *solution sets*. Inequalities $x^2 - 5x - 6 > 0$ and $(x+1)(x-6) > 0$ are equivalent, because they have the same solution set.

equivalent matrices Or row-equivalent matrices. If a matrix could be transformed to another matrix by a finite number of elementary row operations, then the matrices are called equivalent.

equivalent system Two systems of equations that have the same solution sets. See also *equivalent equations*

equivalent vectors Two vectors that have the same direction and *length*.

error Any time an exact value (of a number, function, series, integral, etc.) is substituted by an *approximate value*, an error occurs. Errors could be estimated in absolute terms (such as the difference between exact and approximate values) or in relative terms (as a proportion of error to the exact value). For different specific error estimates see the entries following this one on error estimates of series and integrals.

error estimate for alternating series Assume that the series $\sum_{n=1}^{\infty} a_n$ is *alternating* and is convergent. Denote by S its sum and by S_N its Nth *partial sum*. Then $|S - S_N| < |a_{N+1}|$. See also alternating series test and alternating series estimation theorem.

error estimate for the Midpoint rule Assume that we wish to calculate numerically the integral $\int_a^b f(x)dx$ using the Midpoint rule and denote by M_n the nth approximation using that rule. Then, if $|f''(x)| \le M$ for all $a \le x \le b$, then

$$\left| \int_a^b f(x)dx - T_n \right| \le \frac{M(b-a)^3}{24n^2}.$$

This estimate is very similar to the estimate for the *Trapezoidal rule*.

error estimate for Simpson's rule Assume that we wish to calculate numerically the integral $\int_a^b f(x)dx$ using the Simpson's rule and denote by S_n the nth approximation using that rule. Then, if $|f^{(4)}(x)| \le M$ for all $a \le x \le b$, then

$$\left| \int_a^b f(x)dx - S_n \right| \le \frac{M(b-a)^5}{180n^4}.$$

error estimate for the Trapezoidal rule Assume that we wish to calculate numerically the integral $\int_a^b f(x)dx$ using the Trapezoidal rule and denote by T_n the nth approximation using that rule. Then, if $|f''(x)| \le M$ for all $a \le x \le b$, then

$$\left| \int_a^b f(x)dx - T_n \right| \le \frac{M(b-a)^3}{12n^2}.$$

This estimate is very similar to the estimate for the *Midpoint rule*.

estimate of the sum of a series The sums of most of the series are difficult or even impossible to calculate. Estimates serve as a substitute for exact value and are good enough if the *error* is small. See also *error estimate for alternating series*.

Euclidean distance Distance, generated by the *Euclidean norm*. For two vectors $\mathbf{x} = (x_1, x_2, \cdots, x_n), \mathbf{y} = (y_1, y_2, \cdots, y_n)$ in *Euclidean space*, their distance is defined to be the norm of their difference: $dist(\mathbf{x}, y) = ||\mathbf{x} - y||$.

Euclidean inner product For two vectors in the *Euclidean space* $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, and $\mathbf{y} = (y_1, y_2, \cdots, y_n)$, the quantity

$$\mathbf{x} \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

It is the same as *dot product*. See also *scalar product*.

Euclidean norm For a vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ in *Euclidean space* the number $||\mathbf{x}|| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. This number is non-negative and it is zero if and only if the vector is the zero vector. Also called *magnitude* or *length* of the vector.

Euclidean space The *vector space* of all *vectors* with n coordinates, where the addition and *scalar multiplication* operations are defined in the usual way:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n),$$

$$\alpha \mathbf{u} = (\alpha u_1, \alpha u_n, \cdots, \alpha u_n)$$

for any two vectors $\mathbf{u}, \mathbf{v}$ and any real number α. Additionally, the *inner product* is defined in this space as the *dot product* (or, which is the same, *Euclidean inner product*). This, on its turn, induces the *norm* in the space, which defines *Euclidean distance*. This special vector space is now called Euclidean space and is commonly denoted by R^n, where n is any positive integer.

Euler's constant The limit

$$\gamma = \lim_{n \to \infty} (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n),$$

which exists and is finite. The value of γ is approximately 0.577212 but it is not known if the number is rational or not. See also harmonic series.

Euler equation See Cauchy-Euler equation.

Euler's formula The formula

$$e^{i\theta} = \cos\theta + i \sin\theta,$$

where $i = \sqrt{-1}$ is the imaginary unit. This formula is true also when the real number θ is substituted by any complex number z. In the particular case when $\theta = \pi$, we get the famous formula $e^{i\pi} = -1$ which connects all the fundamental numbers of mathematics.

Euler method For approximation of solutions of *initial value problems*. The idea behind this method is to substitute the solutions of the problem (which usually are difficult or even impossible to find) by a *linear function*, which is the tangent line to this solution at some point. By this reason the method is also called tangent line method. In detail, assume we are solving the problem

$$y' = f(t, y) \qquad y(t_0) = y_0$$

in some interval. As a first approximation we tale the linear function

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

which passes through the point (t_0, y_0) and hence, coincides with the solution at that particular point. Next, we choose a point t_1 not too far from t_0 to assure that the error is small and construct another line

$$y = y_1 + f(t_1, y_1)(t - t_1),$$

where $y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$, is the value found from the first line. Proceeding this way, we will have a series of equations

$$y = y_n + f(t_n, y_n)(t - t_n),$$

each of which is given on the interval $[t_n, t_{n+1}]$. These approximate solutions converge to the exact solution when the lengths of intervals approach zero and under certain conditions on function f.

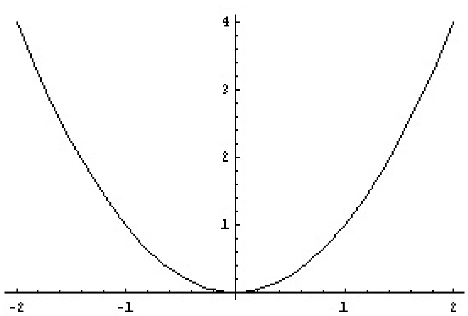
Euler-Fourier formulas Also called Fourier formulas. These formulas represent *Fourier coefficients* of the function by certain integral formulas involving the function itself. Let $f(x)$ be defined on some interval $[-L, L]$. Then its Fourier coefficients are given by

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx, \; n = 0, 1, 2...$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \; n = 1, 2, 3...$$

evaluation The term is used in mathematics in different situations, most commonly when we want to find the value of a function. In this case, to evaluate a function means to find its value for certain value of the independent variable(s). Example: To evaluate the function $f(x) = 3x^4 - 2x^2 + x - 4$ at the point $x = 2$ means to calculate the value of f at that point: $f(2) = 3 \cdot 2^4 - 2 \cdot 2^2 + 2 - 4 = 38$.

even function A function $f(x)$ of real variable, that satisfies the condition $f(-x) = f(x)$. This condition means that the graph of the function is symmetric with respect to the y-axes. The functions $f(x) = \cos x$ and $f(x) = x^2$ are examples of even functions.



even permutations A permutation that is the result of even number of transpositions.

event A term used in Probability theory and Statistics to indicate some kind of outcome for the (usually random) variable. Tossing a coin and getting Tail or Head are two different events. If an event cannot be split into any smaller events, then it is called *simple event.* The collection of all simple events forms the event space. Events can be *dependent, independent, mutually exclusive.* See corresponding entries for more information.

exact equations Let $M(x, y)$ and $N(x, y)$ be functions on some rectangle $R = [a, b] \times [c, d]$. The first order ordinary differential equation

$$M(x, y) + N(x, y)y' = 0$$

is called exact, if there exists a differentiable function $F(x, y)$ on R, such that

$$\frac{\partial F}{\partial x} = M, \quad \frac{\partial F}{\partial y} = N. \qquad (1)$$

Every solution of this equation is given by the implicit function $F(x, y) = c,\ c = \textit{constant}$. The necessary and sufficient condition for the equation being exact is

$$M_y(x, y) = N_x(x, y). \qquad (2)$$

Example: To solve the equation $(3x^2 - 2xy + 2)dx + (6y^2 - x^2 + 3)dy = 0$ we first easily check that the above conditions are satisfied. Next, to find the function F we integrate the first of equations (1) with respect to the variable x and get

$$F(x, y) = \int (3x^2 - 2xy + 2)dx$$

$$= x^3 - x^2 y + 2x + g(y),$$

where $g(y)$ is an arbitrary function of the variable y. To satisfy the second of the equations (1) we differentiate this function by y and equate it to the function N:

$$-x^2 + g'(y) = 6y^2 - x^2 + 3,$$

from where, $g'(y) = 6y^2 + 3$, or $g(y) = 2y^3 + 3y$ (no need of arbitrary constant). Finally, solution of the equation will be given by the implicit equation

$$F(x, y) = x^3 - x^2 y + 2x + 2y^3 + 3y = C.$$

existence and uniqueness theorems For differential equations. Statements, that contain conditions under which the given differential equation has a solution and that solution is unique. Depending on type of equation (order, coefficients, homogeneous or not, boundary or initial conditions, etc) the existence and uniqueness conditions differ from each other. For example, if the equation is of the first order, then one boundary or initial condition is usually enough, while for the second order equations at least two conditions are necessary. Similar observations are valid for systems of equations too. Example:
Theorem. Assume we have the equation

$$y'' + p(t)y' + q(t)y = g(t)$$

with *initial conditions* $y(t_0) = y_0,\ y'(t_0) = y'_0$ where all the functions p, q and g are continuous on some interval I, containing point t_0. Then there exists exactly one solution $y = \phi(t)$ of this problem and the solution exists on all the interval I.

expanded form A term used in various situations. For a *polynomial* $p(x) = (x-2)(x+5)$ the expanded form will be $p(x) = x^2 + 3x - 10$. The term is used also for *determinants*, when expanded form means the form when it is expanded as a sum of *cofactors* or *minors*.

expansion by minors One of the methods of calculating the value of a *determinant*. See corresponding entry.

expansion operator A linear operator that "expands" given vector. In the most general setting, an expansion operator with factor $k > 0$ is a operator that makes the *norm* of a vector $\mathbf{x}$ be equal $k||\mathbf{x}||$. Example: Operator given by a matrix

$$\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

transforms any vector (x, y) into vector $k(x, y)$. See also contraction operator, dilation operator.

expected value For a *distribution*, given by the function $f(x)$, the quantity

$$\mu = \int_{-\infty}^{\infty} x f(x) dx,$$

if it is finite. Expected value is the same as the *mean* of the distribution. This quantity could be defined also for distributions of several variables.

experimental study In *statistics*, a study where the researcher controls, modifies, or changes the conditions in order to collect data of the results and effects of these changes. Most of the medical studies are experiments to assess the results of treatment or medication.

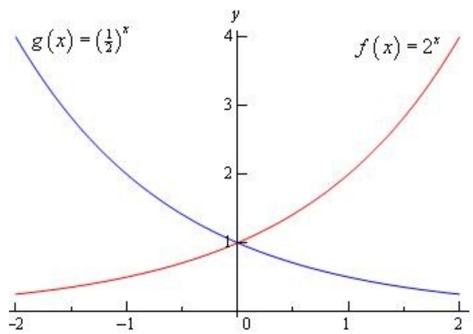
explanatory variable In *statistics*, another name for the independent variable.

exponential decay A functional model that is expressed with the use of *exponential function* with negative exponent:

$$f(t) = Ae^{kt}, \quad k < 0.$$

Many economic, biological and other phenomena have exponential decay model.

exponential function The function $f(x) = a^x$, where $a > 0$, $a \neq 1$, which is defined for all real values of x. The range of the function is $(0, \infty)$.



exponential growth A functional model that is expressed with the use of *exponential function* with positive exponent:

$$f(t) = Ae^{kt}, \quad k > 0.$$

Many economic, biological and other phenomena have exponential growth.

exponential order Functions that have growth or decay comparable with the growth or decay of the exponential function. The $\sinh x$ and $\cosh x$ functions both have exponential order. See hyperbolic functions.

exponentiation One of the main mathematical operations. The exponent of a number b, $b > 0$ (called base) of some power x is denoted by b^x. In the case when x is a *natural number*, this expression is understood as the number b multiplied by itself x times: $b^x = b \cdot b \cdots b$. For exponents other than natural numbers, this definition is extended in several steps.
(1) By definition $b^0 = 1$ for any $b > 0$.
(2) If n is a positive integer, then $b^{-n} = 1/b^n$ by definition.

(3) For positive integers n, we define the exponent $b^{1/n}$ to be the nth root of the number b, i.e. that is a number that raised to the nth power (which is natural) results in b: $(\sqrt[n]{b})^n = b$.
(4) For any integers n and m the exponent $b^{n/m}$ is defined to be the number $(\sqrt[m]{b})^n = \sqrt[m]{b^n}$.
(5) To define the exponent b^x for *irrational number* x we use limiting process: This power is defined as the limit

$$b^x = \lim_{n\to\infty} b^{c_n},$$

where $\{c_n\}$ is some sequence of rational numbers that approaches x.
In definition of exponents we usually avoid the case $b = 1$ because it results in trivial case. From this definition the following important properties of exponentiation follow:
(a) $a^n \cdot a^m = a^{n+m}$
(b) $(a^n)^m = a^{n\cdot m}$
(c) $(ab)^n = a^n \cdot b^n$
(d) $a^n/a^m = a^{n-m}$.
See also *exponential function*

exponents at the singularity In differential equations. When solving an equation by the power series method and the point is regular singular point, the roots of the corresponding indicial equation are called exponents at the singularity.

expression In mathematics, any combination of mathematical objects (numbers, functions, matrices, transformations, etc.) and operations on them. See also *algebraic expression* and *arithmetic expression.*

extraneous root or solution In solving certain types of equations, some roots, that turn out invalid. This happens in solving rational, radical, logarithmic, and some other types of equations. Example: The equation

$$\sqrt{2x+7} - x = 2$$

has two solutions, $x = 1,\ x = -3$ and the second one is extraneous.

extrapolation A procedure of finding values of functions or data based on some known values of the function or given data. Similar to the notion of *interpolation.* The least square regression line is an example of extrapolation when we predict the values of the data based on a *sample* data.

extreme value *Maximum* or *minimum* values of a function.

extreme value theorem If $f(x)$ is a continuous function on the closed interval $[a, b]$, then there exist points $c,\ d \in [a, b]$ such that $f(c)$ is the minimum and $f(d)$ is the maximum values of f on $[a, b]$.
A very similar statement is true also for functions of two or more variables.

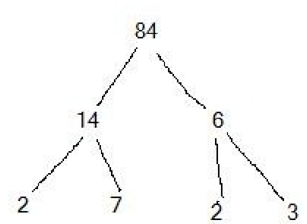
extremum Collective name for *minimum* and *maximum.*

F

F-distribution The distribution of the ratio of variances of two normally distributed populations. This distribution depends on both *degrees of freedom* of that normal distributions, hence there is not one but a family of F-distributions.

factor If a number or function a is written as a product $a = b \cdot c$, then the numbers or functions b and c are called factors. A number or function may have more than two factors. The most common functions we factor are polynomials. See also *factoring.*

factor tree Assume we need to factor a *natural number* into the product of prime factors. Then, by the *fundamental theorem of arithmetic*, the number n could be written as a product $p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m}$. Another way of writing this product is to let it expand vertically, and the resulting presentation is called factor tree. Example: $84 = 2^2 \cdot 3 \cdot 7$ could be written also as a factor tree



factorial of a whole number $n \geq 1$ is defined to be the product $1 \cdot 2 \cdot 3 \cdots n$, denoted by $n!$. By definition, $0! = 1$.

factoring (1) The procedure of representing a number as a product of *factors.* Example: $24 = 3 \cdot 8$. See also *prime factorization.* (2) The procedure of representing a *polynomial* as a product of simpler polynomials. Example: $2x^2+5x-12 = (2x-3)(x+4)$.

family of functions The term is used in cases when instead of one particular function a group (or collection) of functions is present. This fact is usually indicated by the presence of one or more *parameters.* By changing these parameter we get new functions, which, as a rule, do not change the nature of functions. Examples: (1) The family of functions Ax^2+B depends on two parameters and by changing the values of A and B we get different functions but all of them are quadratic. (2) When integrating the function $\cos x$ the result is $\sin x + C$, which is a family of functions depending on one parameter. Change of C creates a new function but it is still a sinusoidal function.

family of solutions When solving differential equations the solution usually depends on one or more *parameters* creating a family of solutions. To have a particular solution to a given equation it is necessary to impose certain restrictions on function, called *initial* or *boundary* conditions. See also *initial value problems, boundary value problems.*

feasible solution Any solution of a *linear programming problem* (a system of linear inequalities) is called feasible solution. The set of all feasible solutions is the feasible region. As a rule, the feasible region is some kind of (possibly infinite) polygon.

Fermat's theorem Let the function $f(x)$ be defined on the interval $[a, b]$. Assume that in some point c inside that interval the function gets its maximum or minimum value. If the function is differentiable at that point, then $f'(c) = 0$.

Fermat's Last Theorem If $n > 2$ is an integer, there are no positive integers x, y, z such that $x^n + y^n = z^n$.
This statement more properly should have been called conjecture, because Fermat did not prove it. It was proven more than 300 years later, in mid 1990s by A.Wiles.

Fibonacci sequence (numbers) The sequence of numbers defined by the following rule: First two terms of the sequence are ones. Every other term is the sum of the previous two terms. First few Fibonacci numbers are: $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$.

field (1) A set of numbers where two algebraic oper-

ations of addition and multiplication are defined and satisfy standard rules (axioms) of that operations. In particular, both operations are *commutative* and *associative* and together they follow *distributive rule*. The most common numeric fields are the fields of *rational numbers, real numbers* and *complex numbers*. The set of integers (or positive integers) is not a field, because reciprocals are not integers.
(2) More precisely, vector field. This notion is defined for vectors on the plane, space, or even higher dimensional spaces. For exact definition see vector field and for specific vector fields see also *conservative vector field, gradient vector field, curl of a vector field*.

finite dimensional vector space A vector space that has only finitely many linearly independent vectors. See *dimension of vector space*.

first derivative test Let $f(x)$ be defined on some interval $[a, b]$ and in some inner point c, $a < c < b$, $f'(c) = 0$. (1) If $f'(x) > 0$ for $c - \delta < x < c$ and $f'(x) < 0$ for $c < x < c + \delta$, then f has local maximum at point c. (2) If $f'(x) < 0$ for $c - \delta < x < c$ and $f'(x) > 0$ for $c < x < c + \delta$, then f has local minimum at point c. (3) If $f(x)$ does not change the sign while passing through point c, then the test cannot give definitive result.

first order differential equation Any differential equation that contains only the first derivative of the unknown function. Depending on the other components of the equation is classified as linear, nonlinear, with constant or variable coefficients, and also some others. A typical first order linear equation with variable coefficients is

$$y' + p(t)y = g(t),$$

because the function y is present in the first degree only and the coefficients p, q are functions, not constants. For specific equations of first degree see exact equations,Bernoulli equation, separable equations.

five number summary In statistics. The five numbers for some data set are the minimum value, the maximum value, the *median*, and the first and third *quartiles*. These five numbers are important for construction of the boxplot.

fixed point of a function Any point $x = a$ in the domain of some function f that satisfies the equation $f(a) = a$. If the function is viewed as a *mapping*, then the fixed points are the ones that do not move during the mapping f. Not all functions have fixed points and other functions may have more than one fixed point. Examples: The function $f(x) = e^x$ does not have any fixed points because the equation $e^x = x$ has no solutions. The function $f(x) = x^2 + x - 4$ has two fixed points $x = \pm 2$, which are the solutions of the equation $x^2 + x - 4 = x$.

flux Let $\mathbf{F}$ be a continuous *vector field* defined on an *oriented surface* S with unit normal $\mathbf{n}$. Then the surface integral of $\mathbf{F}$ over the surface S

$$\int \int_S \mathbf{F} \cdot n \, dS$$

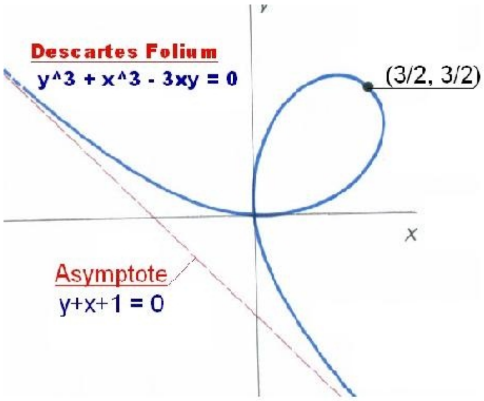
is called the flux of $\mathbf{F}$ across S.

focus of a conic section All three major conic sections have one (parabola) or two (ellipse and hyperbola) points that are important in geometric definitions of that conic sections. For the definitions and details see *ellipse, hyperbola, parabola* and also eccentricity. Plural for the word focus is foci.

foil method of multiplication of two *binomials*. To multiply $(a + b)(c + d)$, we multiply (F) first terms $a \cdot c$, (O) outside terms $a \cdot d$, (I) inside terms $b \cdot c$, (L) last terms $b \cdot d$, and add them up:

$$(a + b)(c + d) = ac + ad + bc + bd.$$

folium of Descartes A plane curve given by the cubic equation $x^3 + y^3 = 3axy$, where a is some real parameter. This curve has a *slant asymptote* given by the equation $x + y + a = 0$.



force One of the most important notions in Physics. By the Newton's second Law, if an object of mass m moves along a line with the position function $s(t)$, then the force on the object is the product of the mass and its acceleration:

$$F = m\frac{d^2s}{dt^2}.$$

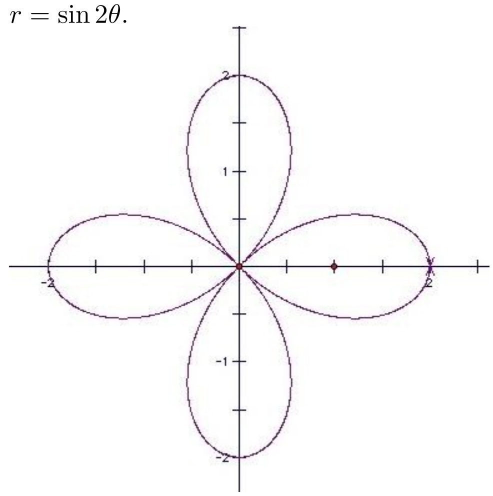
forced vibrations When the vibration of a spring or a pendulum is affected by some additional force $F(t)$, then this vibration obeys the following differential equation:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t).$$

formula An equation that contains two or more variables. As a rule, a formula is deduced to calculate values of one variable if the values of other variable(s) are known. Elementary formulas in most cases are interchangeable in a sense that they could be solved for any other variable. This is not true for more complicated formulas. Here are a few familiar formulas from geometry and algebra. See corresponding entries for more details.
Area of a rectangle: $A = \ell \times w$.
Compound interest: $P = A(1 + r/n)^{nt}$.
Distance formula:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

four-leafed rose The plane curve, given by *polar equation* $r = a \sin 2\theta,\ a > 0$. The curve is called so because it consists of four loops. It is a special case of more general family of polar curves given by the equations $r = a \sin n\theta,\ r = a \cos n\theta$, where n is an arbitrary integer. The graph shows the equation $r = \sin 2\theta$.



Fourier coefficients (1) In the *Fourier series*

$$a_0/2 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

the coefficients $a_n,\ n \geq 0$ and $b_n,\ n \geq 1$.
(2) In a more general setting, if the function $f(x)$ is defined on interval $[0, 1]$, $r(x)$ is some *weight function*, and $\phi_1(x), \phi_2(x), \cdots, \phi_n(x)$ is an *orthonormal set*, then the numbers

$$a_k = \int_0^1 f(x)\phi_i(x)r(x)dx, \quad i = 1, 2, ..., n,$$

are the Fourier coefficients of f.

Fourier series For the given integrable function $f(x)$ on the interval $[0, 2\pi]$, the formal series

$$f(x) = a_0/2 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta),$$

where the coefficients are defined by the formulas

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx,\ n = 0, 1, 2, ...$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx,\ n = 1, 2, 3...$$

Fourier series of a function converges to that function if it satisfies certain conditions, e.g., if it is differentiable. Also called trigonometric series.

fraction Any real number that could be written in the form $\frac{n}{m}$, where n and m are arbitrary real numbers with only one condition that $m \neq 0$. In the particular case when n and m are integers fractions are called *rational numbers*. For operations with fractions see *addition and subtraction of fractions, multiplication of fractions, division of fractions*. See also *complex fractions* and *partial fractions* for special types of fractions.

frequency For trigonometric functions $f(x) = a\sin\omega t$ and $g(x) = a\cos\omega t$, the number $\omega/2\pi$ is the frequency. This number shows how many times the functions complete one full cycle when we move the distance equal to 2π. The reciprocal to the frequency is the *period* of the function.

Fresnel functions The functions

$$S(x) = \int_0^x \sin t^2 dt$$

and

$$C(x) = \int_0^x \cos t^2 dt.$$

The integrals on the interval $[0, \infty)$ are both convergent and are equal to $\sqrt{\pi/8}$.

Frobenius method The method of solving linear differential equations near a regular singular point. See the entry series solution for the details.

Fubini's theorem In its most general form this theorem gives conditions on function of several variables which allow to calculate *multiple integrals* as *iterated integrals*. In the simplest form the theorem for double integrals is formulated as follows: Let f be a continuous function on the rectangle $R = \{(x, y)|a \leq x \leq b,\ c \leq y \leq d\}$. Then

$$\int\int_R f(x,y)dA = \int_a^b \left[\int_c^d f(x,y)dx\right] dy$$

$$= \int_c^d \left[\int_a^b f(x,y)dy\right] dx.$$

Similar formulas hold also for triple or multiple integrals.

function One of the most important mathematical objects. In order to define a function it is necessary first of all to have two sets associated with the function called the *domain* and the *range*. Then we can formally define the function as some relation (mapping, transformation) that translates each element of the domain to exactly one element of the range. If the requirement of having exactly one corresponding element in the range is removed, then instead of function we have a *relation*. Functions can be represented in many different ways.

1) The first, least common form of representation is the verbal representation. As an example we can describe a function as follows: The function translates each value of the variable to its double.

2) The function can be represented also numerically, by a table, matrix, chart, etc. For example, if the domain of the function is the set $\{1, 2, 3, 4, 5\}$ and the range is $\{6, 9, 13, 17\}$ then the function can be given as a list of ordered pairs: $\{(1, 6), (2, 9), (3, 17), (4, 9), (5, 13)\}$, meaning that the function translates the point 1 to the point 6, point 2 to the point 9, etc.

3) The function can have graphical representation when instead of numeric values just the graph is given from where the numeric values may be found. This method of representing functions is not very accurate.

4) The most important form of representation of functions is the algebraic (often also called analytic) form, when an explicit formula is given from which the values of the function can be calculated. The functions represented algebraically, usually written with function notation, which simplifies the process of finding the values at a given point. For example, if a function given with the notation $f(x) = 2x^2 - 3$, then the value of the function at the point $x = -2.7$ will be found by substituting this value for x : $f(-2.7) = 2(-2.7)^2 - 3 = 11.58$.

Functions given algebraically are further categorized as *algebraic and transcendental*. On their turn, alge-

braic functions have subcategories of *polynomial, rational, radical functions* and the transcendental functions can be *exponential, logarithmic, trigonometric* and others. For specific examples see the corresponding entries.
Further, functions could be *increasing, decreasing, even, odd, one-to-one, periodic, continuous, differentiable, integrable* and of many other types.
5) Functions can be represented as series, such as power series or Fourier series. For example, the rational function $f(x) = 1/(1-x)$ on interval $(-1, 1)$ could be represented as

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Similarly, we have a Fourier series representation on interval $(0, \pi)$

$$\frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{\sin nx}{n}.$$

6) Functions could be represented also with integrals. Laplace transform is the most common example of integral representations of functions.
The notion of the function could be extended also to the case of two, three or, more generally, n variables. Also, functions can be *vectors* themselves, or be *vector-valued*. For specific types of function see also *Bessel functions, harmonic functions, implicit functions*.

fundamental matrix spaces For a matrix A, the collective name of the following four spaces: Row space of A, column space of A, the *nullspace* of A, and the nullspace of the *transpose* of the matrix A^T.

fundamental solutions In the most general setting, a (generalized) function ϕ is called the fundamental solution of the partial differential operator L, if it satisfies the equation $L\phi = \delta$, where δ is the Dirac function. For specific partial differential equations we have the following expressions for fundamental solutions:
Heat conduction equation:

$$a^2 u_{xx} = u_t, \quad 0 < x < M,\ t > 0$$

$$u_n(x,t) = e^{-n^2\pi^2 a^2 t/M^2} \sin\frac{n\pi x}{M},\ n \geq 1.$$

The wave equation:

$$a^2 u_{xx} = u_{tt}, \quad 0 < x < M,\ t > 0$$

$$u_n(x,t) = \sin\frac{n\pi x}{M} \cos\frac{n\pi at}{M},\ n \geq 1.$$

Laplace's equation:

$$u_{xx} + u_{yy} = 0, \quad 0 < x < a,\ 0 < y < b$$

$$u_n(x,t) = \sinh\frac{n\pi x}{b} \sin\frac{n\pi at}{b},\ n \geq 1.$$

fundamental theorem of algebra If $p(x)$ is a polynomial of degree $n > 0$, then the corresponding polynomial equation has at least one solution in the field of complex numbers.
As a result of this theorem it follows that any polynomial of degree $n > 0$ has exactly n roots (real or complex), if we count the roots with their multiplicities.

fundamental theorem of arithmetic Any positive integer n could be factored in a product of powers of prime numbers: $n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_m^{k_m}$. This presentation is unique except possible order of prime factors. Example: $2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$.

fundamental theorem of calculus Let f be a Riemann integrable function on the interval $[a, b]$ and let F be the indefinite integral (antiderivative) of the function f on $(a, b) :\ F'(x) = f(x)$. Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

This theorem allows to calculate *definite integrals* without elaborate process of partitioning the intervals, constructing *Riemann sums* and passing to the limit. It also combines two main branches of *calculus*: the differential calculus and integral calculus.
Example: Since $(\sin x)' = \cos x$,

$$\int_0^\pi \cos x dx = \sin\pi - \sin 0 = 0.$$

fundamental trigonometric identities The identities

$$\cot x = \frac{1}{\tan x},\ \sec x = \frac{1}{\cos x},\ \csc x = \frac{1}{\sin x}$$

and

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}.$$

See also *Pythagorean identities, cofunction identities,addition and subtraction formulas, double-angle, half-angle formulas, sum-to-product, product-to-sum formulas.*

G

gamma function For the real positive variable x, the function defined by the formula

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

This function is the generalization of the factorial in the sense that $\Gamma(n+1) = n!$. Gamma function can also be defined for complex values of argument.

Gaussian elimination One of the most effective methods of solving systems of linear algebraic equations. The idea of the method consists of eliminating more and more variables in successive equations, and to arrive to the situation, when the last equation has only one unknown. This will allow to find the value of that variable and substitute in the previous equation that has only two variables, hence finding the value of the second unknown. Continuing this *back substitution*, we will be able to solve the system. During this process, only elementary row operations are performed, which guarantees, that the resulting system is equivalent to the original one, i.e., has the same solution set. Also, the procedure may be applied directly to the system of equations or the corresponding *augmented matrix*. We will demonstrate the second approach. Let

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

be our system of n equations with n variables $x_1, x_2, \cdots, x_n$ and let

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & b_n \end{pmatrix}$$

be the corresponding augmented matrix. First, we divide first row by a_{11} and get 1 in the first position of the first row. Next, we multiply first row by $-a_{21}$ and add to the second row, to get zero on the first position of the second row. After dividing second row by $a_{22} - a_{12}a_{21}/a_1$, we will have the following format

$$\begin{pmatrix} 1 & a'_{12} & \dots & a'_{1n} & b'_1 \\ 0 & 1 & \dots & a'_{2n} & b'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{pmatrix}$$

Now, continuing, we can eliminate first element of the third row by using the first row, second element of the third row by using the second row, and then dividing by the value of the resulting third element, we will make it one. As a final result we will find the matrix in *upper triangular* form, with all entries on the *diagonal* being 1's:

$$\begin{pmatrix} 1 & a'_{12} & \dots & a'_{1n} & b'_1 \\ 0 & 1 & \dots & a'_{2n} & b'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b'_n \end{pmatrix}$$

This triangular matrix corresponds to triangular system of equations, that is solved by the *back substitution* method. This same method works also for more general case of $m \times n$ systems but does not result in unique solution.
Example: To solve the system of equations

$$2x + y - 4z = 3$$

$$x - 2y + 3z = 4$$

$$-3x + 4y - z = -2$$

we form its augmented matrix

$$\begin{pmatrix} 2 & 1 & -4 & 3 \\ 1 & -2 & 3 & 4 \\ -3 & 4 & -1 & -2 \end{pmatrix}.$$

As the first step we interchange first and second rows to get a 1 on the left upper corner (notation: $R_1 \leftrightarrow R_2$), then multiply first row by -2 and add to the second row ($-2R_1 + R_2 \to R_2$) to get 0 on the left column of the second row. Similarly, doing the operation $3R_1 + R_3 \to R_3$ will eliminate the first entry on the third row. Now, the matrix has the form

$$\begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 5 & -10 & -5 \\ 0 & -2 & 8 & 10 \end{pmatrix}.$$

On the next two steps we multiply second row by 1/5 and the third row by -1/3 to get 1's in the middle column. Finally, after doing the operations $-R_2 + R_3 \to R_3$ and multiplying the last row by -1/2, we arrive to

$$\begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

This matrix corresponds to the system of equations

$$x - 2y + 3z = 4$$

$$y - 2z = -1$$

$$z = 2,$$

which could be solved by *back substitution* method and gives the solution set $(4, 3, 1)$.

Gauss-Jordan elimination The development of *Gaussian elimination* method. Here, the method does not stop at triangular matrix but makes it diagonal. Assume, that as a result of Gaussian elimination, our augmented matrix is reduced to the following one:

$$\begin{pmatrix} 1 & a_{12} & \dots & a_{1n} & b_1 \\ 0 & 1 & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-1n} & b_{n-1} \\ 0 & 0 & \dots & 1 & b_n \end{pmatrix}$$

Now, if we multiply the last row by $-a_{n-1n}$ and add it to the previous row, the nth term of that row will become zero. The only non-zero elements of that row will remain the $(n-1)$st element (which is on the diagonal and equals one) and the last entry, which will be equal now $b_{n-1} - a_{n-1n}b_n$. Next, using both bottom rows, we can eliminate two elements on the $(n-2)$nd row, except the diagonal element (equals one) and the $(n+1)$st entry. Continuing this way up we eventually arrive to the matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & b'_1 \\ 0 & 1 & \dots & 0 & b'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & b'_{n-1} \\ 0 & 0 & \dots & 1 & b_n \end{pmatrix},$$

which, in turn, is equivalent to the system of equations

$$x_1 = b'_1,\ x_2 = b'_2, \cdots, x_{n-1} = b'_{n-1},\ x_n = b_n.$$

Hence, this last system provides the solution immediately.

As an example we will continue solution of the system from the previous entry. To solve the system

$$2x + y - 4z = 3$$

$$x - 2y + 3z = 4$$

$$-3x + 4y - z = -2$$

we apply the *Gaussian elimination* method and arrive to the matrix

$$\begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Continuing by the Gauss-Jordan method, we first do the operation $2R_3 + R_2 \to R_2$ to eliminate third entry on the second row and then the operation $-3R_3 + R_1 \to R_1$ eliminating the third entry on the first row. The result is the matrix

$$\begin{pmatrix} 1 & -2 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

On the last step we add two times row two to the first row and finally

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

which corresponds to the system

$$x = 4, \quad y = 3, \quad z = 2,$$

already solved for all variables.

general multiplication rule for probabilities Let A and B be two *events*, not necessarily independent. Then the *multiplication rule for probabilities* generalizes as follows:

$$P(A \cap B) = P(A) \cdot P(B|A),$$

where $P(B|A)$ is the conditional probability of the event B given A. Another notation for this rule is $P(A \text{ and } B) = P(A) \cdot P(B|A)$.

general solution of linear equations For differential equations. Let $Lf(x)$ be some *linear differential operator*. The general solution of the equation $Lf(x) = g(x)$ is the solution that presents all possible solutions of that equation. Hence, general solution is not a function but a family of functions, depending on some numeric parameters. If the equation is of the nth degree, then, as a rule, there are n parameters in this family of functions. For linear equations of any degree, the general solution of the non-homogeneous equation $Lf(x) = g(x)$ is the sum of the general solution of the homogeneous equation $Lf(x) = 0$ and any particular solution of the original equation. Similar definition and facts are true also for systems of linear equations.

general solution of trigonometric equations The solution of any trigonometric equation that provides all possible answers. Example: The equation

$$\cos 4x \sin x = 0$$

splits into two equations $\sin x = 0$ and $\cos 4x = 0$. The first one has the general solution $x = \pi n,\ n = 0, \pm 1, \pm 2...$ and the second one gives $x = \pi/8 + \pi n/4,\ n = 0, \pm 1, \pm 2...$ All the solutions of the original equation are described by the set $S = \{x | x = \pi n,\ x = \pi/8 + \pi n/4,\ n = 0, \pm 1, \pm 2...\}$.

general term An expression used in various situations. Most commonly is used for polynomials to indicate the term $a_j x^j$ with some $j = 0, 1, 2, ...$

generators of the cone See the definition of the *cone*.

geometric distribution One of the *probability distributions*, where the probability that k trials are needed to get one success is

$$P(X = k) = (1 - p)^{k-1}p$$

and p is the probability of success on each trial.

geometric mean For given set of $n \geq 1$ non-negative numbers, the quantity

$$\sqrt[n]{a_1 \cdot a_2 \cdots a_n}.$$

geometric sequence or progression A sequence of real numbers (finite or infinite), where the ratio of each term (starting from the second) to the previous term is the same constant. The general term of these sequences is given by the formula

$$a_n = a_1 r^{n-1},$$

where a_1 is the first term of the sequence and r is the ratio of the sequence.

geometric series A series $\sum_{n=1}^{\infty} a_n$, where the terms form a *geometric sequence*. If the *ratio* r of that sequence satisfies $|r| < 1$, then the series is convergent and the sum is given by the formula

$$\sum_{n=1}^{\infty} a_n = \frac{a_1}{1 - r}.$$

For an arbitrary ratio r, the partial sums of the series are given by the formula ($m \geq 1$ is an arbitrary integer)

$$\sum_{n=1}^{m} a_n = \frac{a_1(1 - r^m)}{1 - r}.$$

golden ratio Two numbers, a and b, form the golden ratio, if the ratio of the larger number to smaller, say a/b, is equal to the ratio of their sum to the larger number $(a + b)/b$. Simple calculations show that the exact value of that ratio is $(1 + \sqrt{5})/2$, which is approximately 1.618033988...

golden ratio rectangle A rectangle, where the length and width form the *golden ration*, i.e., $l/w = (l + w)/l = (1 + \sqrt{5})/2$.

gradient For a function f of several real variables, the gradient is a *vector*, which coordinates are the partial derivatives of the function:

$$\bigtriangledown f(x_1, x_2, \cdots, x_n) = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}).$$

The gradient vector points to the direction of largest growth of the function.

gradient vector field For functions of three variables. Let $f = f(x, y, z)$ be a scalar function of three variables. Then the gradient of the function written in the vector form

$$\bigtriangledown f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

is called gradient vector field. Similar definition is valid for functions of any number of independent variables. See also curl of a vector field, divergence of a vector field.

Gram-Schmidt process A method of *orthonormalization* of a basis of a vector space. Suppose $\{\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}\}$ is a basis of the vector space V. We start with any vector of the system, say $\mathbf{v_1}$ and call it $\mathbf{u_1}$ To get the second vector, we take $\mathbf{v_2}$ and subtract from it its *projection* on the vector $\mathbf{v_1}$:

$$\mathbf{u_2} = \mathbf{v_2} - \mathbf{proj_{u_1} v_2}.$$

This vector is orthogonal to $\mathbf{u_1}$. In the next step, we take the third vector in our basis and subtract from it its projections on first two vectors:

$$\mathbf{u_3} = \mathbf{v_3} - \mathbf{proj_{u_1} v_3} - \mathbf{proj_{u_2} v_3}.$$

The vector $\mathbf{u_3}$ is now orthogonal to both $\mathbf{u_1}$ and $\mathbf{u_2}$. Continuing this process, we get a new system of vectors $\{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_n}\}$, which is orthogonal. In the last step of this process we just normalize this last system by dividing each of the vectors by their *length (magnitude, norm)*. The new system

$$\mathbf{w_1} = \frac{\mathbf{u_1}}{||\mathbf{u_1}||}, \mathbf{w_2} = \frac{\mathbf{u_2}}{||\mathbf{u_2}||}, \cdots, \mathbf{w_n} = \frac{\mathbf{u_n}}{||\mathbf{u_n}||}$$

is now orthonormal, because each vector has length one.

The Gram-Schmidt process works also for infinite bases.

graph of an equation Let $y = f(x)$ be some equation defined on an arbitrary interval $[a, b]$ (finite or infinite). The graph of the equation is the geometric place of all points $(x, f(x))$ on the *Cartesian plane*, when the variable x varies over its interval of existence. Every graph represents some kind of *curve*, depending on the nature of the equation. Graphs are helpful visual representations of equations.

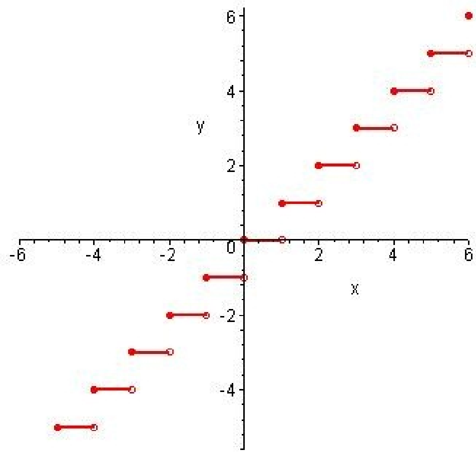
graph of a function Let $y = f(x)$ be some function defined on an arbitrary interval $[a, b]$ (finite or infinite). The graph of the function is the geometric place of all points $(x, f(x))$ on the *Cartesian plane*, when the variable x varies over its interval of existence. Every graph represents some kind of *curve*, depending on the nature of the function f. The difference between the graph of a function and the *graph of an equation* is that for the function to each point on the interval $[a, b]$ there is only one corresponding point on the plane. Graphs are helpful visual representations of functions.

greater than An inequality, stating that one quantity is greater than the other, with notation $A > B$. If the quantities are *real numbers*, then this statement means that A is to the right of B on the *number line*. See also *less than*.

greater than or equal to An *inequality*, stating that one quantity is greater than the other, or equal to it, with notation $A \geq B$. If the quantities are *real numbers*, then this statement means that A is to the right of B on the *number line*, or the two points coincide. See also *less than or equal to*.

greatest common factor (GCF) For two positive integers m and n, the largest integer, that divides both of that numbers, denoted by $gcf(m, n)$. Example: $gcf(12, 18) = 6$. Also called greatest common divisor (GCD).

greatest integer function For the given real number x the function $[[x]]$ is defined to be the greatest integer such that $n \leq x$. This function has *jump discontinuity* at each integer point and is constant for all values between two integers.



greatest lower bound Also called infimum. If a function f or a sequence a_n are bounded below, then "biggest" of all possible lower bounds is the greatest lower bound. Example: The sequence $a_n = 1 + 1/n$ is bounded below by any number $M \leq 1$ but not by any number greater than 1. The number 1 is the greatest lower bound. See also *least upper bound*.

Green's identity Let $u(x, y)$ and $v(x, y)$ be two times continuously differentiable functions in the *finitely simple* region D with the boundary C. Denote by dA the area measure in D, by ds the arc measure on C and by $\mathbf{n}$ the unit outer normal to the boundary C. Then

$$\int\int_D (u\Delta v - v\Delta u)dA = \int_C (u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}})ds.$$

Here Δ denotes the *Laplace* operator.

Green's identity Let $M(x, y)$ and $N(x, y)$ be continuously differentiable functions in the *finitely simple* region D and let C be the boundary of D. Then

$$\int_C Mdx + Ndy = \int\int_D (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy.$$

This theorem is one of the possible generalizations of the Fundamental Theorem of Calculus to double integrals, because it allows to calculate integrals over plane domains as line integrals over the boundaries of that domains.

grouping symbols The symbols ( ),[ ],{ } and sometimes others, that are used to group numbers, functions, expressions, etc. Primarily are used to indicate and determine the order of operations to be performed. In many cases some other symbols also play the role of grouping symbols. For example, in expressions $5+|3-6|-4$ or $3-\sqrt{4+5}$ the operations inside the absolute value and square root signs should be done before any other operations.

growth rate Also could be called rate of growth. For a given variable quantity $s(t)$ the rate with which it grows. Mathematically it is the same as the derivative, i.e., growth rate of s is $s'(t)$.

H

half-angle formulas In trigonometry, the formulas

$$\cos^2\frac{\theta}{2}=\frac{1+\cos\theta}{2}$$

$$\sin^2\frac{\theta}{2}=\frac{1-\cos\theta}{2}$$

$$\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}=\frac{\sin\theta}{1+\cos\theta}.$$

Similar formulas exist for other trigonometric functions but are rarely used.

half-life The period of time required for a quantity (usually a radioactive material) to reduce to its half. If the initial amount of the quantity is A_0, then the amount at the time t is given by $A(t)=A_0e^{-\lambda t}$. Half-life depends on material (and is reflected in constant λ) and can vary from a few seconds to many thousand years.

half-open intervals Intervals of the form $[a,b)$ or $(a,b]$, where a and b are any real numbers. Notation means that one of the ends of the interval is included and the other one is not.

half-plane One of the four possible configurations on the *Cartesian* plane: $\{(x,y)|x>0\}$, $\{(x,y)|x<0\}$, $\{(x,y)|y>0\}$, $\{(x,y)|y<0\}$. More generally, the solution of a linear inequality $ax+by<c$ (or other three alternative forms of that inequality involving the symbols $>$, $\geq$, or $\leq$), where a,b,c are real numbers.

half space In three dimensional *Euclidean space*, the solution of any inequality of the form $Ax+By+Cz<D$ (or other three alternative forms of that inequality), where A,B,C,D are real numbers. This is the same, as to say that half space is the set of all points, that lie on one side of some plane. This notion is possible to extend to higher dimensional spaces.

harmonic function Any function u of two or more variables in a given domain, that satisfies Laplace's equation. In the case of two variables the equation has the form

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0.$$

harmonic motion or simple harmonic motion. Describes the *periodic* movement of some object about the *equilibrium point* that can be given by one of the following equations:

$$s(t) = a\cos\omega t,\ s(t) = a\sin\omega t.$$

Here s is the displacement of the object at the time t, a is the amplitude of motion and $\omega/2\pi$ is the *frequency* of *oscillation*.

harmonic series The numeric series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

This series is divergent and according to Euler's formula, its partial sums go to infinity as the natural logarithmic function.

heat conduction equation In Partial Differential Equations: for a function of two spacial variables x, y, and the time variable t, the equation

$$\frac{\partial u(x,y,t)}{\partial t} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}.$$

In the simpler case of one spacial variable x and time variable t the equation has the form $a^2 u_{xx} = u_t$.

Heaviside function The function

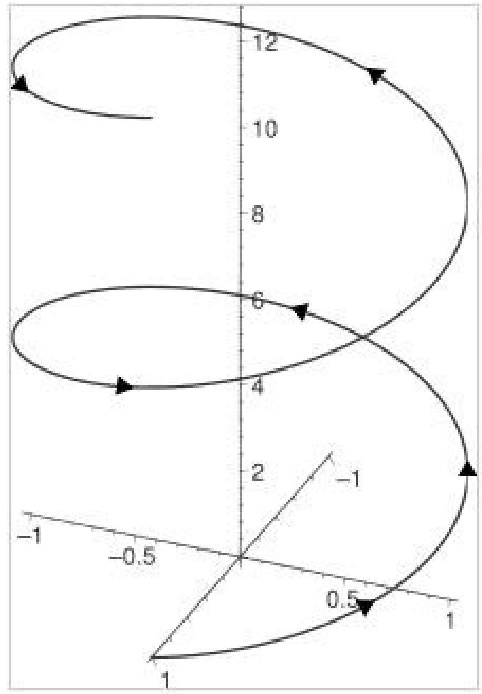
$$f(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

This function plays important role in physics and electrical engineering. See also Laplace transforms of special functions.

helicoid A 3-dimensional surface given by the equations

$$x = r\cos t,\ y = r\sin t,\ z = \alpha t,$$

where both r and α range from $-\infty$ to ∞. For any fixed values of that parameters, helicoid becomes a *helix*.



helix A curve in 3-dimensional space given by the equations

$$x = \cos t,\ y = \sin t,\ z = t.$$

Here t is a real parameter. As it increases, the point (x, y, z) traces a helix about the z-axis.

Heron's formula For the area of a triangle with the sides, equal to a, b and c. Denote $s = (a+b+c)/2$. Then the area A is given by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$

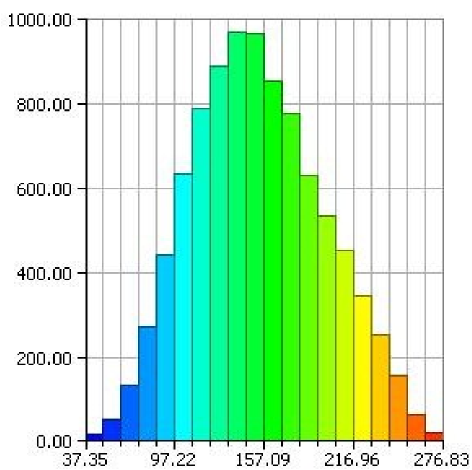
higher order derivatives Since the derivative of a function is also a function, we can calculate the derivative of that function. If it exists, then that will be the second derivative of the function:

$$f''(x) = (f'(x))' = \frac{d^2 f}{dx^2}.$$

Similarly, the third derivative, $f'''(x) = \frac{d^3 f}{dx^3}$, the fourth, $f^{(4)}(x) = f^{\text{iv}}(x) = \frac{d^4 f}{dx^4}$, and the nth derivative $f^{(n)}(x) = \frac{d^n f}{dx^n}$ are defined, if they exist.

Hilbert space A linear vector space, equipped with inner product and the notion of distance associated with that inner product. The term almost exclusively applies to infinite dimensional vector spaces.

histogram In *statistics*, a histogram is a graphical display of frequencies. A histogram is the graphical version of a table which shows what proportion of cases fall into each of several specified categories. The histogram differs from a bar chart in that it is the area of the bar that denotes the value, not the height. The categories are usually specified as non-overlapping intervals of some variable. Another difference between histograms and bar graphs is that the bars in histograms are adjacent, not separated as for bar graphs.



homogeneous algebraic equations Systems of linear algebraic equations, where all of the right sides are zero. Homogeneous systems are always *consistent*, because they always have the trivial solution (i.e. all zero solution). Homogeneous systems may have also non-trivial solutions, if any of the equations is a linear combination of multiple other equations.

homogeneous differential equation A differential equation where only the unknown function and its derivatives are present, possibly multiplied by some (variable or constant) coefficients. Examples: The equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

is homogeneous, but the equation

$$y'(t) - 2y(t) = 3x^2$$

is not. In this last case equation is called *non-homogeneous.*
Homogeneous equations are also classified by their order (first, second,etc.), by the type of coefficients (constant or variable), by linearity and so on. In cases when we have several equations with several unknown functions, we get system of homogeneous equations.

Hooke's law In physics, this law states that the force necessary to hold a spring stretched x units beyond the equilibrium point is proportional to the displacement x, if its small. Symbolically, $f(x) = kx$, where k is a positive number called spring constant.

horizontal asymptote See asymptote.

horizontal axis The x-axis in *Cartesian* coordinate system. It is given by the equation $y = 0$.

horizontal line A line, that is parallel to the x-axis in *Cartesian* coordinate system. Any horizontal line is given by the equation $y = a$, where a is any real number.

horizontal line test If a function $f(x)$ is given graphically, and any horizontal line intersects that graph only once, then the function is *one-to-one.* If even at one point this condition is violated then the function cannot be one-to-one. This test allows to determine if the given function has an *inverse* or not but is not accurate.

hyperbola One of the three main *conic sections.* Geometrically, a hyperbola is the location (locus) of all points in a plane, the difference of whose distances from two fixed points in the plane is a positive constant. The two points, from which we measure distances, are called *foci* (plural for *focus*). Equivalently, the hyperbola could be described as the result of cutting *double cone* by a plane parallel to its axis. Hyperbola is a *smooth curve* with two *branches.* Alternative geometric definition could be given with the

use of *eccentricity*. See corresponding definition.
Algebraically, the general equation of a hyperbola is given by the equation $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$, where A, B, C, D, E, F are real constants and $A\cdot C<0$. In the case, when the foci are located on one of the coordinate axes and center coincides with the origin, this equation could be transformed into *standard form*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

where the constants a, b have special meaning: $c=\sqrt{a^2-b^2}$ is the distance from origin to a focus. Moreover, the intersection points of hyperbola with one of the axes are called *vertices* of hyperbola and their distance from origin is equal to a. The line connecting two foci is called *transverse axis* and the perpendicular line (passing trough origin) is the *conjugate axis*. Each hyperbola of this form has two *slant asymptotes* given by the formulas $y=\pm\frac{b}{a}x$.

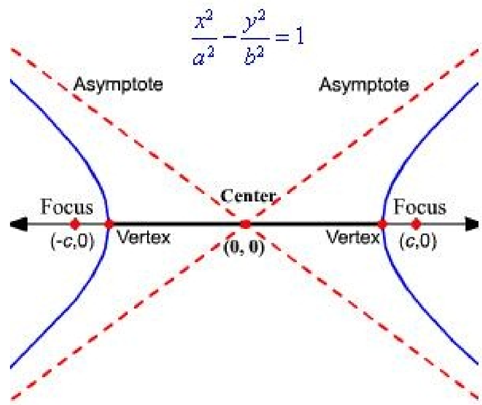
In the more general case,when the center of hyperbola is located at some point (h,k) but the transverse and conjugate axes still parallel to coordinate axes, the standard equation becomes

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1,$$

and the equations of the slant asymptotes are $y=k\pm\frac{b}{a}(x-h)$. In all cases the *eccentricity* is given by the formula $e=c/a$.
Both of the above cases happen when in the general equation the term Bxy is missing. In the case $B\neq 0$ the result is still a hyperbola which is the result of rotation of one of the previous simpler cases about the center of hyperbola.
The hyperbola could also be given by its polar equation:

$$r=\frac{de}{1\pm e\cos\theta}\quad\text{or}\quad r=\frac{de}{1\pm e\sin\theta},$$

where $d>0$ and $e>1$ is the *eccentricity* of the hyperbola.



hyperbolic functions The functions

$$\sinh x=\frac{e^x-e^{-x}}{2},\quad \cosh x=\frac{e^x+e^{-x}}{2},$$

$$\tanh x=\frac{e^x-e^{-x}}{e^x+e^{-x}},\quad \coth x=\frac{e^x+e^{-x}}{e^x-e^{-x}},$$

Hyperbolic secant and cosecant functions are defined as reciprocals of hyperbolic cosine and sine functions, respectively. The $\sinh x$ and $\cosh x$ functions are the most used with a very limited use for other four hyperbolic functions.

hyperbolic identities Identities for hyperbolic functions, analogous to corresponding trigonometric identities. Here the equivalent of the *Pythagorean* identity is the following:

$$\cosh^2 x-\sinh^2 x=1.$$

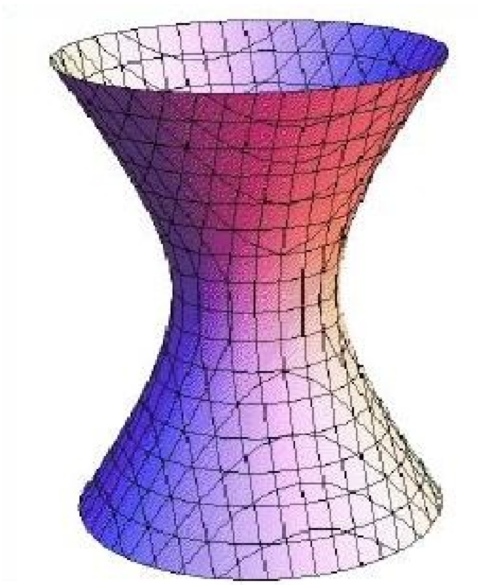
Another example is the formula

$$\sinh(x+y)=\sinh x\cosh y+\cosh x\sinh y.$$

hyperbolic paraboloid The three dimensional surface given by the formula

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}.$$

hyperbolic substitution A method of calculation of *indefinite integrals*, when the independent variable is substituted by one of the *hyperbolic functions*. Substitutions are used to evaluate indefinite integrals when ready to use formulas are not available. See also trigonometric substitutions and substitution method.



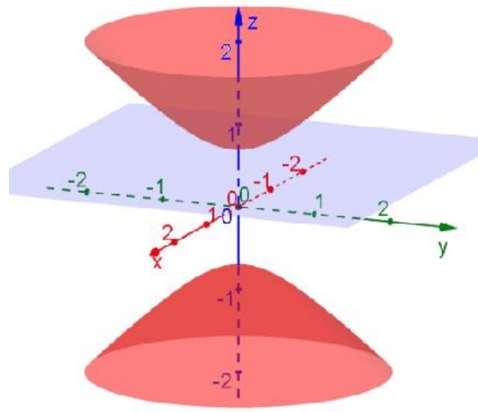
hyperboloid (1) Hyperboloid of one sheet is given by the formula

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

and represents a surface that is connected.
(2) Hyperboloid of two sheets is given by

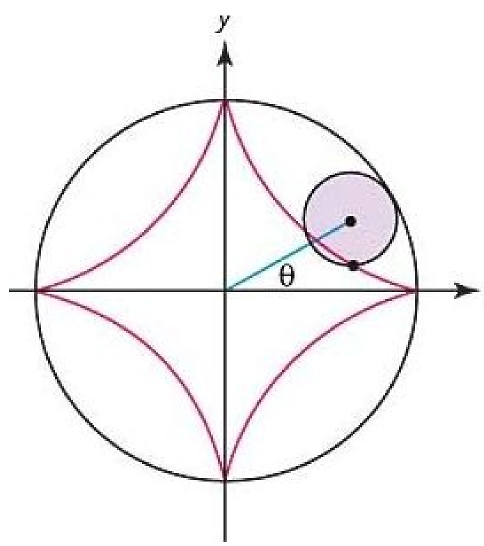
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

and consists of two pieces.



hypersphere Usually, the name of the n-dimensional *sphere* given by the equation $x_1^2 + x_2^2 + \cdots + x_n^2 = r^2$, where r is the radius.

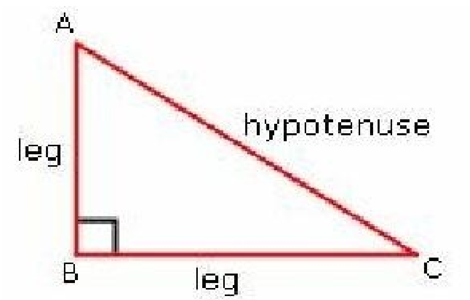
hypervolume Usually, volume in n-dimensional *Euclidean* space.



hypocycloid Geometrically, a curve that is traced by a point on a circle that rolls inside a bigger circle. Depending on ratio of radiuses of the circles, the result is a different curve. If that ratio is 1/4, the result is called *astroid* (see picture above). In general, a hypercycloid with $n+1$ cusps is given by parametric equations

$$x = \cos\theta + \frac{1}{n}\cos n\theta,\ y = \sin\theta - \frac{1}{n}\sin n\theta.$$

hypotenuse In the *right triangle*, the side, opposite to the *right angle*. The other two sides are called *legs*.



hypothesis A statement or proposition, which validity is not known but which is very likely to be true based on experimental or theoretical observations. One of the best known is the so-called Riemann hypothesis.

hypothesis testing In Statistics. the procedure of testing a working assuption (hypothesis) using a set of values from a sample. For details see null hypothesis and alternative hypothesis.

I

identity (1) An *equality*, containing constants and/or variables, that is true for all values of the variables, whenever the expression makes sense. Examples are: $(a+b)^2 = a^2+2ab+b^2$, $\sin^2\theta+\cos^2\theta = 1$, $1+\tan^2 x = \sec^2 x$. The first two identities are valid for all values of a, b or θ, and the last identity is valid for all values of $x \neq \pi/2+\pi k$, $k = 0, \pm1, \pm2, ...$, because the functions involved in the identity are not defined for that values of argument. See also *trigonometric identities.*
(2) A number, that added to other numbers or multiplied by other numbers does not change them. See *additive identity, multiplicative identity.*

identities, trigonometric See trigonometric identities.

identity matrix The square matrix where all the entries are zero except of the main diagonal, where all the entries are ones. Example:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

When we multiply the identity matrix by any other matrix from the right or left, the given matrix does not change: $IA = AI = A$.

identity function The function $f(x) = x$ which leaves each point in its place.

identity transformation A transformation T of a vector space V that leaves all elements of the space unchanged: For any $\mathbf{v} \in V$, $T\mathbf{v} = \mathbf{v}$. Also called identity operator.

image of a linear transformation. If $T : V \to W$ is a linear transformation from one vector space to another, then for any *vector* $\mathbf{u} \in V$ the vector $T(\mathbf{u}) \in W$ is its image. The set of all images is called the image of the transformation T.

imaginary axis The vertical axis in the *complex plane*. When the complex plane is identified with the

Cartesian plane, imaginary axis is the same as y-*axis*.

imaginary number A number of the form ai, where a is a real number and $i = \sqrt{-1}$ is the *imaginary unit*.

imaginary unit The number $i = \sqrt{-1}$. This number is not real, because $i^2 = -1$. See also *complex numbers*.

implicit differentiation When a function is given implicitly (see *implicit function*), then it is still possible to calculate the derivative of that function. Let the function is given by the equation $F(x, y) = 0$ and assume that F has *partial derivatives* with respect to both variables. In that case, we will be able to calculate the derivative of y with respect to the variable x if we view y as a function of x and use the chain rule. Example: If the implicit function is given by the equation

$$\arctan y/x = x,$$

then differentiation by x gives

$$\frac{\frac{dy}{dx}x - y}{x^2 + y^2} = 1$$

and, after the simplification,

$$\frac{dy}{dx} = \frac{x^2 + y^2 + y}{x},$$

which itself is an implicit function.

implicit function A function of two variables, that is not explicitly solved for any of the variables. This relation can be expressed as an equation $F(x, y) = 0$ with some function F. Examples are $x^2 + y^2 - 1 = 0,\ 2\sin^3 x - 3xy = 0$. Similar definition holds for functions of several variables too. Let $x_1, \cdots, x_n, y$ be $(n + 1)$ variables and F represents some functions of that variables. Then the equation $F(x_1, \cdots, x_n, y) = 0$ is an implicit function if the equation is not solved for any of the variables.

Implicit Function Theorem gives conditions under which a function defined implicitly, can be solved for the dependent variable. We give the simplest version for the case of two variables. Let $F(x, y) = 0$ be an implicit function defined on some *disk*, that contains a point (a, b). Assume that $F(a, b) = 0$, $\frac{\partial F}{\partial y}(a, b) \neq 0$, and both partial derivatives of F are continuous on the disk. Then the equation $F(x, y) = 0$ defines y as an explicit function of x near the point (a, b) and the derivative of that function is given by the formula

$$\frac{dy}{dx} = -\left(\frac{\partial F}{\partial x}\right) / \left(\frac{\partial F}{\partial y}\right).$$

Similar statements are valid for functions of three or more variables also.

implicit solution A solution of an (usually differential) equation which is given as an implicit function $F(x, y) = 0$.

implied domain When the domain of some function is not stated explicitly but is clear from the definition of the function, then it is said that the domain is implied.

improper fraction A fraction, where the *numerator* is greater than or is equal to the *denominator*. Examples: $\frac{7}{5},\ \frac{25}{25}$.

improper integral An integral, where either one or both integration limits are infinite, or the function is unbounded on the bounded interval.
(1) Let the function $f(x)$ be defined on the interval $[a, \infty)$ and assume that the integral $\int_a^A f(x)dx$ exists for any given number $A \geq a$. Then the limit of that integral (finite or infinite) is the integral of f over the interval $[a, \infty)$:

$$\int_a^\infty f(x)dx = \lim_{A\to\infty} \int_a^A f(x)dx.$$

In case this limit is finite, the function is called integrable on the given interval. The improper integrals over the intervals $(-\infty, b]$ and $(-\infty, \infty)$ are defined similarly.
(2) Let $f(x)$ be defined on a bounded interval $[a, b]$ but be unbounded there, for example, as we approach the right endpoint b. Assume also, that on each interval $[a, b - \delta]$ the function is bounded and the integral $\int_a^{b-\delta} f(x)dx$ exists. Then the limit of that integral (finite or infinite) is the integral of f over the inter-

val $[a, b]$:

$$\int_a^b f(x)dx = \lim_{\delta \to 0} \int_a^{b-\delta} f(x)dx.$$

In case the limit is finite, the function is called integrable on the given interval. Improper integrals, when the left end-point or some inside point of the interval are the points where f is not bounded, are defined in a similar manner.

improper integral comparison theorem For improper integrals on infinite intervals.
Assume that f and g are continuous functions defined on some interval $[a, \infty)$ and $f(x) \geq g(x)$ on that interval.
(1) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)\, dx$ is convergent;
(2) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)\, dx$ is divergent.

impulse function Let $g(t)$ be a function defined for all real values t, with the condition that it is zero outside some "small" interval $[t_0 - \tau, t_0 + \tau]$ and with finite integral on that interval. These kind of functions represent some type of impulse (mechanical, electrical, or other) and its integral $I(\tau) = \int_{-\infty}^\infty g(t)dt$ is the total impulse of the force $g(t)$. The limiting case of these type of impulse functions as the interval of its existence goes to zero, is the Dirac function.

impulse response Assume we have some differential equation. We can take the equation

$$ay'' + by' + c = g(t)$$

for an example. If in the right-hand side the function g is the *Dirac function* $\delta(t)$, then the solution of this equation is called impulse response (to the impulse δ).

inconsistent linear system A system of linear equations that has no solution. The system is *consistent*, if it has at least one set of solutions. Example: The system of equations

$$x - 2y = 5$$

$$2x - 4y = 3$$

is inconsistent because it has no solutions.

increasing/decreasing test Also called increasing/decreasing theorem.
(1) If the function is defined on some interval I and $f'(x) > 0$ for all points on that interval, then the function is *increasing*.
(2) If the function is defined on some interval I and $f'(x) < 0$ for all points on that interval, then the function is *decreasing*.

increasing function The function $f(x)$ defined on some interval I is increasing, if for any two points $x_1, x_2 \in I,\ x_1 < x_2$, we have $f(x_1) < f(x_2)$.

increasing sequence A sequence of real numbers $\{a_n\},\ n \geq 1$, is increasing, if $a_m > a_k$ whenever $m > k$.

indefinite integral For a given function $f(x)$ on some interval, indefinite integral is any function $F(x)$, which *derivative* is that function:

$$F'(x) = f(x).$$

In fact, indefinite integral is not one specific function, but a *family* of functions. Along with F, any function of the form $F(x) + C$, where C is an arbitrary *constant*, is also indefinite integral of f. For indefinite integral of a function f we have the notation $\int f(x)dx$. Examples:

$$\int \sin x dx = -\cos x + C,$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln(x + \sqrt{a^2 + x^2}) + C.$$

Indefinite integral is also called *antiderivative*.

indefinite matrix A *square matrix* for which the expression $\mathbf{x}^T A\mathbf{x}$ takes both positive and negative values. Here $\mathbf{x}$ is an arbitrary vector and $\mathbf{x}^T$ is its transpose. See also positive definite, negative definite matrices.

independence of path Let $\mathbf{F}$ be some vector field in two or three dimensional space and assume that some piecewise smooth curve (path) C is given by the *vector-function* $\mathbf{r}(t)$. The integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is said to be independent of path, if its value depends on initial and terminal points of the path C only.
Theorem 1. The integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if for any *closed path* C, $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$. Additionally, the next theorem establishes when this situation takes place.
Theorem 2. Suppose $\mathbf{F}$ is a continuous vector field in some open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a *conservative vector field*, i.e., there exists a function f such that $\mathbf{F} = \bigtriangledown f$.

independent events Two or more *events* are independent, if none of them affects the outcome of any of the others. If A and B are the two events and P denotes the probability, then the relation of independence could be expressed in the form $P(A \textit{ and } B) = P(A)P(B)$, or, equivalently, $P(A \cap B) = P(A)P(B)$. Independence could be extended also to three or more events and the definition will be similar.

independent observations For an observation independence has the same meaning as for an event. Similarly, when collecting *samples* we get independent samples if no one depends on any other collected sample. See also *simple random sample.*

independent variable In an equation $y = f(x)$ the variable x. The variable y is called *dependent variable.* A function of several variables $y = f(x_1, x_2, \cdots, x_n)$ has n independent variables $x_1, x_2, \cdots, x_n$.

indeterminate forms of limits Let $f(x)$, $g(x)$ be two functions defined around the point $x = c$ (but not necessarily at that point). When calculating the limit

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

it is possible that $\lim_{x \to c} f(x), g(x) = 0$ or $\lim_{x \to c} f(x), g(x) = \infty$. In these cases we say that the limit is indeterminate form $0/0$ or ∞/∞ respectively. Also we may have $0 \cdot \infty$ indeterminate form while calculating the limit of the product of f and g. In other situations, indeterminate forms $\infty - \infty$, 1^∞ and 0^0 also may arise. In many cases these limits could be calculated by the l'Hospital rule.

index of summation In the *sigma notation* (or *summation notation*) presentation of a finite or infinite *series*

$$\sum_{n=1}^{m} a_n$$

the parameter n. As a rule, index of summation takes either *integer* or *whole* values.

indicial equation An *algebraic equation*, associated with some differential equations. This equations come out in cases when we solve differential equation by series method around regular singular point. Example: Assume we have the equation

$$2x^2y'' - xy' + (1 + x)y = 0.$$

The point $x = 0$ is a regular singular point and to solve this equation we need to solve corresponding indicial equation

$$2r(r - 1) - r + 1 = 0,$$

which produces two roots $r = 1,\ 1/2$. Now, two solutions of the original equation are found by multiplying x^r by certain *power series*:

$$y_1 = x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n x^n}{(2n+1)!}\right],$$

$$y_2 = x^{1/2}\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^n x^n}{(2n)!}\right].$$

induction See mathematical induction.

induction hypothesis One of the steps in the method of *mathematical induction*, when we assume that certain statement is true for the value of the natural number k and then proceed to prove the statement for the next integer $k + 1$.

inequality A statement that one quantity does not equal another quantity. The statement could be written in one of the following four possible forms: $A < B$, $A \leq B$, $A > B$, or $A \geq B$. Se also *quadratic inequality.*

inference, statistical See statistical inference.

infinite discontinuity A function is said to have infinite discontinuity at some point $x = c$, if one or both of the *left-hand* or *right-hand limits* is either ∞ or $-\infty$. The function may or may not be defined at that point.

infinite interval An *interval*, where one or both of the endpoints are infinite. The only three possible types of infinite intervals are $(-\infty, a)$, (b, ∞) and $(-\infty, \infty)$.

infinite limit (1) A *sequence* of numbers has infinite limit if the members of the sequence grow indefinitely. In precise definition, $\{a_n\}$, $n = 1, 2, 3, ...$, has infinite limit, if for any given positive number M there exists an integer $N > 1$, such that $a_n \geq M$ whenever $n \geq N$.
(2) A function $f(x)$ has infinite limit at some point $x = c$, if for any positive M, there exist $\delta > 0$ such that $|f(x)| > M$ whenever $|x - c| < \delta$. This last definition requires modification if the point c is infinite. In that case the definition is very similar to the definition of infinite sequence.

infinite non-repeating decimal A decimal representation of a real number, where no one *digit* or group of digits repeat. This kind of decimal numbers cannot represent integers or rational numbers, so all of them are irrational. Well known examples are the decimal approximations of numbers $\sqrt{2} = 1.414213562...$ and $\pi = 3.141592654....$

infinite repeating decimal A decimal representation of a real number, where one *digit* or a group of digits repeat over and over again. Each repeating infinite decimal represents a rational number. The opposite is also true: each rational number can be represented as a finite or repeating infinite decimal. We use a bar over the group of repeating digits to indicate the fact that they are repeating. Examples: $\frac{1}{3} = 0.333.... = 0.\overline{3}$, $\frac{2}{7} = 0.\overline{285714}$.

infinite sequence A sequence $\{a_n\}$, where the index n takes unbounded number of values (usually, whole numbers or integers).

infinite series A series where infinitely many *terms* are added. See also convergent series and divergent series.

inflection point Let $y = f(x)$ be a continuous function on some interval. A point P on its *graph* is called inflection point, if the graph changes its *concavity* when we pass through that point. The function $y = x^3$ has inflection point at $x = 0$, because to the left of that point it is concave down and to the right it is concave up.

influential observation Or influential point. In a scatterplot one (or some) points are called influential, if removing them results in a very different regression line. Usually, all the points that are very far from the majority of points in either horizontal or vertical direction (or both) are influential.

initial conditions For ordinary differential equations. The conditions at the *initial point* to assure that the equation has unique solution.

initial point If the ordinary differential equation is given on some interval $[a, b]$ (most commonly $b = \infty$), then the point a is called initial point.

initial point of vector A vector defined as directed line interval, has two endpoints on this interval. The point where the vector starts, is the initial point. Because only the direction and length are important for a vector, we place the initial point at the origin of the coordinate system.

initial side of angle Of two rays forming an angle in *standard position* the one that coincides with the positive x-axis. See also *angle*.

initial value problem Also called Cauchy problem. For ordinary differential equations. The equation along with conditions on solution at the initial point. To solve initial value problem means to find solution to the equation that also satisfies given conditions at the initial point. Initial conditions are chosen to assure uniqueness of the solution. Example: The equation

$$y'' + p(t)y' + q(t)y = 0$$

for $0 \leq t < \infty$, with conditions $y(0) = 0$ and $y'(0) = 1$ is an initial value problem.

inner product for vectors in real vector space is defined as a real number satisfying the following con-

ditions: Let $\mathbf{x}, \mathbf{y}$ be two vectors. Then
(1) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$; (2) For any real a, $\langle a\mathbf{x}, y \rangle = a\langle \mathbf{x}, y \rangle$; (3) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = 0$.
Inner product is also called *scalar product*. In the special case when the space is the *Euclidean space*, and the inner product is defined as the product of *coordinates* of the vectors, then it is also called *dot product*. Inner product for *complex vector space* is defined in a similar manner but requires involvement of complex numbers and some additional conditions.

instability of critical point For solutions of differential equations. Let

$$\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y$$

be the *logistic equation*. The functions $y = K$ and $y = 0$ are two constant solutions of this equation. The other solutions of this equation with the *initial condition* $y(0) = y_0$ are given by the formula

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}. \qquad (1)$$

The first solution $y = K$ is stable because all the solutions (1) approach that solution as $t \to \infty$. On the other hand the solution $y = 0$ is unstable because the solutions (1) do not approach it. This solution is also called unstable critical point.

instantaneous rate of change If a quantity is changing during the time period $[t_1, t_2]$, then the average rate of change is its net change divided by the elapsed time $t_2 - t_1$. When the interval $[t_1, t_2]$ is getting smaller and smaller (eventually becoming a single point), the resulting value is the instantaneous rate of change of the quantity. This is one of the main problems that helped to create differential calculus.

instantaneous velocity The *instantaneous rate of change* for velocity.

integers The set of integers is the union of *natural* (or counting) numbers $\{1, 2, 3, \cdots\}$, their opposites $\{-1, -2, -3, \cdots\}$ and the number 0.

integral One of the most important notions of *calculus*. Has two main meanings. See definite integral and indefinite integral for basic definitions. For integrals in other settings see also *double integral, line integral, surface integral* and many other definitions related to integrals and integration.

integral curves The graphical presentations of general solutions of differential equations, that depend on one or more *integration constants*.

integral equation An equation where the unknown function appears under the integral sign. Two most important types of integral equations are

$$f(x) = \int_a^b K(x,t)\phi(t)dt$$

and

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\phi(t)dt.$$

Here f and K are given known functions and ϕ is the unknown one. These equations are called Fredholm equations of the first and second type respectively.

integral test Gives convergence criteria for positive series. Assume we can find a positive, continuous, and decreasing function defined on interval $[1, \infty)$, such that $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the integral $\int_1^{\infty} f(x)dx$ is convergent.

integral transform A transform which is given with an integral formula. An integral transform has the form

$$Tf(x) = \int_a^b K(x,t)f(t)dt,$$

where a and b are any numbers (including infinity) and $K(x,t)$ is some (usually integrable) function of two variables, called *kernel of integral transform*. In this formula the function f is regarded as a variable or "vector" and integral transforms are always linear. One of the most common integral transforms is the Laplace transform.

integrand The function which is getting integrated. In expression

$$\int \frac{e^{-x^2}}{2 + x^2}dx$$

the function $f(x) = \frac{e^{-x^2}}{2+x^2}$ is the integrand.

integrating factor Assume we have a general first order linear differential equation

$$y' + p(t)y = g(t)$$

with some given functions p and g. In order to find the *general solution* of this equation, we multiply all terms by some positive function $\mu(t)$, called integrating factor, which makes the left side equal to the derivative of the function $y(x)\mu(x)$. This function can be chosen to be equal $\mu(t) = \exp\int p(t)dt$ and the general solution of the equation is given now in the form

$$y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)ds + c\right].$$

integration The procedure of calculating *definite* or *indefinite* integrals. Unlike differentiation, where the derivative of any analytically given function could be found by applying differentiation rules, integration process is much harder and for many functions there are no simple indefinite integrals. For different methods of finding indefinite integrals see *integration by partial fractions, integration by parts, integration by substitution, by trigonometric substitution, term-by-term integration.* For approximate integration of definite integrals see *approximate integration, numerical integration, Simpson's rule, trapezoidal rule.*

integration by partial fractions A method of *integration* of rational functions. Let $f(x) = P(x)/Q(x)$ be a *proper* rational function. Then it could be decomposed into the sum of partial fractions of one of the following forms:

$$\frac{A}{(ax+b)^j}, \ \frac{Ax+B}{(ax^2+bx+c)^k},$$

where in the second case the denominator is *irreducible* over real numbers. Now, integration of the integral $\int f(x)dx$ is reduced to integration of simpler fractions. Examples:

$$\int \frac{x-3}{x^2-3x+2}dx = \int \left(\frac{2}{x-1} - \frac{1}{x-2}\right) dx$$

$$= 2\ln|x-1| - \ln|x-2| + C.$$

$$\int \frac{12}{x^4-x^3-2x^2}dx$$

$$= \int \left(\frac{3}{x} - \frac{6}{x^2} + \frac{1}{x-2} - \frac{4}{x+1}\right) dx$$

$$= 3\ln|x| + \frac{6}{x} + \ln|x-2| - 4\ln|x+1| + C.$$

$$2\int \frac{x^3+1}{(x^2+1)^2}dx$$

$$= 2\int \left(\frac{x}{x^2+1} - \frac{x}{(x^2+1)^2} + \frac{1}{(x^2+1)^2}\right) dx$$

$$= \ln(x^2+1) + \frac{1}{x^2+1} + \frac{x}{x^2+1} + \tan^{-1}x + C.$$

integration by parts One of the *integration* methods. Let u and v be differentiable functions. Then

$$\int udv = uv - \int vdu$$

and the identical formula holds for the *definite integrals*:

$$\int_a^b udv = uv|_a^b - \int_a^b vdu.$$

This formula allows to calculate many integrals that otherwise were impossible to handle. Example: With $u = \ln x, \ v = x$,

$$\int \ln x \, dx = x\ln x - \int dx = x\ln x - x + C.$$

integration by substitution One of the *integration* methods. Let $f(x)$ and $u = g(x)$ be two functions an u be also *differentiable.* Then

$$\int f(g(x))g'(x)\, dx = \int f(u)\, du.$$

This formula allows to calculate certain integrals, that have special form. Example: To calculate the integral $\int \sqrt{2x+1}\,dx$, we notice, that with substitution $u = 2x+1,\ du = 2dx$, and

$$\int \sqrt{2x+1}\,dx = \frac{1}{2}\int \sqrt{u}du$$

$$= \frac{1}{3}u^{3/2} + C = \frac{1}{3}\sqrt{2x+1} + C.$$

integration by trigonometric substitution A method of evaluating integrals containing expressions of the type $\sqrt{a^2 \pm x^2}$. In that case the substitutions $x = a\sin\theta,\ x = a\tan\theta$ or $x = a\sec\theta$ often result in simpler integrals to evaluate. Examples: (1) To evaluate the integral

$$\int \frac{dx}{(\sqrt{1-x^2})^3},$$

we substitute $x = \sin\theta,\ dx = \cos\theta d\theta$ and get

$$\int \frac{dx}{(\sqrt{1-x^2})^3} = \int \frac{\cos\theta d\theta}{(\cos\theta)^3} = \tan\theta + C,$$

and, returning to original variable, the answer transfers to $x/\sqrt{1-x^2} + C$.
(2) To evaluate the integral $\int \frac{1}{\sqrt{4+x^2}}dx$, we use the substitution $x = 2\tan\theta\ dx = 2\sec^2\theta d\theta$ and see that the integral is transformed to

$$\int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left(\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right) + C.$$

integro-differential equation An equation, where the unknown function is both integrated and differentiated. Example:

$$\phi''(x) = f(x) + \int_a^b K(x,t)\phi(t)dt,$$

where f and K are given functions and ϕ is to be determined from this equation.

intercepts The intersections of the *graph* of a function with the coordinate axes. For $y = f(x)$, the y-intercept is the point, when $x = 0$ and the x-intercept(s) is the point, when $y = 0$. Example: To find the y-intercept of the function $y = x^2 - 4x + 3$, we plug-in $x = 0$ and find $y = 3$ and the point is $(0, 6)$. To find the x-intercepts, we plug-in $y = 0$ and solve the equation $x^2 - 4x + 3 = 0$, which has solutions $x = 1, 3$ and the intercepts are $(1, 0), (3, 0)$.

intermediate value theorem Let $f(x)$ be a continuous function on the closed interval $[a, b]$. Assume that L is a number between the values $f(a)$ and $f(b)$ of the function at the endpoints. Then there exists at least one point c such that $a \le c \le b$ and $f(c) = L$. This theorem helps sometimes to find the x-intercepts of functions. For example, if $f(a) < 0$ and $f(b) > 0$ then the theorem means that there is at least one x-intercept inside the interval $[a, b]$.

interpolation If some points are given on the plane, then any *curve* passing through that points is called interpolating curve. If that curve represents the graph of some polynomial, then it is called interpolating polynomial. In the most general case we may fix some points of the given function f creating a set of ordered pairs $\{(x_1, f(x_1)), (x_2, f(x_2)), ..., (x_n, f(x_n))\}$ and the interpolating polynomial $p(x)$ will represent an approximation to the function f which additionally coincides with given function at all prescribed points. Depending on the degree of the polynomial the interpolation will be linear, quadratic, cubic, etc. See also extrapolation.

intersection A point where two *curves* coincide. If the curves are given by equations $y = f(x)$ and $y = g(x)$, then they intersect, if for some point $x = c,\ f(c) = g(c)$. When a curve intersects with one of the *coordinate axes*, then the intersection point is called *intercept*.

intersection of sets If A and B are two sets of some objects, then their intersection $A \cap B$ is defined to be the set of all elements that belong to both A and B. Example: If $A = \{1, 2, 3, 4, 5, 7\}$ and $B = \{2, 4, 6, 8\}$ then $A \cap B = \{2, 4\}$. In case when two sets have no common elements we say that their intersection is the *empty set*.

interval A subset of the real line with the property that for any two points in that subset, all the points between that two are also included. Intervals could be closed (include both endpoints), open (do not include endpoints), or half-open (include only one of endpoints). The four possible configurations are: $[a, b]$, (a, b), $[a, b)$, $(a, b]$. In this definition one or both endpoints could be infinite. See *infinite interval.*

interval of convergence For a *power series*

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

the interval $(c-r, c+r)$, where the seris converges. The number r is called *radius of convergence* and may be zero, some positive number, or infinity. Some series also converge at the endpoints, hence the interval becomes closed. Example: The series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

has the half-open interval $[-1, 1)$ as its interval of convergence, because when $x = 1$, it results in diverging *harmonic series* and for $x = -1$ it is the converging *alternating harmonic series.*

invariant Something, that does not change when certain transformation is performed. Examples: For the function $f(x) = x^2$ the point $x = 1$ is invariant, because $f(1) = 1$. For any linear transformation T in any *vector space* V, the origin is invariant, because $T(\mathbf{0}) = \mathbf{0}$. In a more general setting, the area of a plane geometric figure is invariant when we transform this figure using only translation, rotation and reflection operations.

inverse function For function $f(x)$ of one real variable with *domain* D, the function $g(x)$ is its inverse, if for all appropriate values of the variable

$$g(f(x)) = f(g(x)) = x.$$

The inverse of the function f is denoted by f^{-1}. Usual sufficient condition for existence of the inverse is the condition of being *one-to-one.* Familiar examples of pairs of inverses are: $f(x) = 2x$, $f^{-1}(x) = 1/2x$; $f(x) = x^3$, $f^{-1}(x) = \sqrt[3]{x}$; $f(x) = 2^x$, $f^{-1}(x) = \log_2 x$.

inverse Laplace transform The transformation, that translates the *Laplace transform* of some function back to the same function. Symbolically,

$$\mathcal{L}^{-1}\mathcal{L}f(t) = f(t).$$

The inverse Laplace transform exists under certain conditions on the original function and is unique. Analytically it is given by a formula involving complex variable integration. A list of inverse Laplace formulas could be deduced from direct Laplace transform formulas working backwards. See the corresponding entry.

inverse matrix For a given square matrix A the matrix B, that satisfies the conditions $AB = BA = I$, where I is the *identity matrix.* Inverse matrix is denoted by A^{-1} and it exists if and only if the *determinant* of the matrix A is not zero. There is a simple formula for calculation of inverses of 2×2 matrices. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the given matrix, then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For general $n \times n$ matrices the inversion formula uses cofactors and is given by

$$A^{-1} = \frac{1}{det(A)} (C_{ij})^T,$$

where $(C_{ij})^T$ is the transpose of the matrix where element on $ij-th$ position is the corresponding cofactor. This formula is very hard to use in practice, so instead a version of Gauss-Jordan elimination method is used. Example: If

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 3 & 4 & 5 \end{pmatrix},$$

we form a 3×6 matrix by adjoining the identity matrix to A:

$$\begin{pmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 3 & 3 & 0 & 1 & 0 \\ 3 & 4 & 5 & 0 & 0 & 1 \end{pmatrix}$$

and apply Gauss-Jordan method. The result will be another 3×6 matrix

$$\begin{pmatrix} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{pmatrix}$$

and the right half of this matrix is the inverse of the given matrix A:

$$A^{-1} = \begin{pmatrix} -3 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{pmatrix}.$$

inverse transformation Let $T : V \to W$ be a linear transformation from one linear vector space to another. The inverse of this transformation, denoted by T^{-1}, is another linear transformation (if it exists), that transforms W to V and satisfies the conditions $T \circ T^{-1} = T^{-1} \circ T = I$, where I is the identity transformation and "$\circ$" denotes *composition of transformations*. Sufficient condition of existence of the inverse transformation is that T be *one-to-one transformation*.

inverse trigonometric functions See *arcsine, arccosine* and *arctangent functions*.

inverse variation The name of many possible relations between two variables. The most common are:(1) y varies inversely with x means that there is a real number $k \neq 0$, such that $y = k/x$; (2) y varies inversely with x^2 means $y = k/x^2$; (3) y varies inversely with x^3 means $y = k/x^3$. There are many other possibilities but rarely used. Compare also with *direct variation* and *joint variation*.

inversion (1) The procedure of finding the inverse of a given object: number, function, matrix, etc.
(2) In a permutation, whenever a larger number precedes a smaller one, we call it an inversion. See also transposition.

invertible matrix A matrix such that the *inverse matrix* exists.

involute A *parametric curve* given by the equations

$$x = r(\cos\theta + \theta\sin\theta), \quad y = r(\sin\theta - \theta\cos\theta).$$

irrational number Any real number that is not *rational*. Irrational numbers cannot be written as *fractions*, as *terminating decimals* or as *non-terminating, repeating decimals*. Common examples of irrational numbers are $\sqrt{2},\ \pi,\ e$.

irreducible polynomial The polynomial is irreducible over a number field, if it cannot be factored into linear factors with coefficients from that field. A polynomial which is irreducible over some number field, may be *reducible* over other number field.
(1) The polynomial $x^2 - 2$ is irreducible over the field of rational numbers, but it is reducible over the field of real numbers because $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$.
(2) The polynomial $x^2 + 4$ is irreducible over rationals and reals but it is reducible over complex numbers because $x^2 + 4 = (x + 2i)(x - 2i)$. By the *Fundamental Theorem of Algebra*, any polynomial with rational coefficients is reducible over the field of complex numbers.

irregular singular point For differential equations. Consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

and its power series solutions. If for the point $x = x_0$, $P(x_0) \neq 0$, then that point is called *ordinary* and series solution is possible near that point. If $P(x_0) = 0$, then the point is *singular*. Additionally, the series solution is still possible if the singular point is *regular*, which means that the limits

$$\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)} \quad \text{and} \quad \lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$$

are finite. If any of these conditions is violated, then the point x_0 is called irregular singular point.

irrotational vector field A *vector field* $\mathbf{F}$ is called irrotational at some point P, if at that point $\text{curl}\mathbf{F} = 0$.

isobars A contour line of equal or constant pressure on a graph.

isosceles triangle A triangle that has two equal sides. Isosceles triangles necessarily also have two equal angles.

iterate In the process of *successive approximation*, the result on each intermediate step is called iterate.

iterated integral Let $f(x,y)$ be a function defined on some rectangle $R=[a,b]\times[c,d]$. We may integrate this function with respect of the second variable and get a new function of one variable

$$F(x)=\int_c^d f(x,y)dy.$$

Now, if we integrate this resulting function with rexpect to the first variable, then we will get

$$\int_a^b F(x)dx=\int_a^b\left[\int_c^d f(x,y)dy\right]dx.$$

The integral on the right side is called iterated integral. Similar definitions hold for functions of three or more variables. Fubini's theorem uses iterated integral in calculations of *multiple integrals.*

J

Jacobian Assume that the variables $x,\ y,\ z$ are expressed with the help of other three variables $u,\ v,\ w$ and the change of variable functions $x=x(u,v,w),\ y=y(u,v,w),\ z=z(u,v,w)$ are continuously differentiable. Then the matrix

$$J=\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{pmatrix}$$

is called the Jacobian matrix of the transformation. The importance of this matrix is that the *determinant* of J plays important role in calculating integrals by change of variable. The following theorem holds: Let the function $f(x,y,z)$ be defined and integrable in some region G and let the change of variable formulas $x=x(u,v,w),\ y=y(u,v,w),\ z=z(u,v,w)$ transform G to some other region G'. Then

$$\int\int\int_G f(x,y,z)\ dxdydz$$

$$=\int\int\int_{G'} f(u,v,w)\ \det(J)\ dudvdw.$$

joint variation The name of many possible relations between three or more variables. The most common are:(1) z varies directly with x and y means that there is a real number $k\neq 0$, such that $z=kxy$; (2) z varies directly with x and inversely with y means $z=k\frac{x}{y}$; (3) z varies directly with x^2 and inversely with y means $z=k\frac{x^2}{y}$. There are many other possibilities including more variables. See also *inverse variation* and *direct variation.*

Jordan curve Let $I=[a,b]$ be an interval and the functions $x=f(t)$ and $y=g(t)$ are defined on that interval. These equations define a plane parametric curve. This curve is called Jordan curve if: (1) The functions $f,\ g$ are continuous; (2) the curve is closed, meaning that $f(a)=f(b),\ g(a)=g(b)$; (3) The curve

is simple, i.e. $f(t_1) \neq f(t_2)$ except the endpoints, and similarly $g(t_1) \neq g(t_2)$.
Jordan curves in the space are defined in the same way.

Jordan decomposition See *Jordan form of matrix.*

Jordan domain A plane region that is bounded by a *Jordan curve.*

Jordan form of matrix A block-diagonal square matrix, that could be written in the form

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & J_n \end{pmatrix},$$

where each J_k is itself a square matrix. The *diagonal* elements of this block-matrix are all the same, the sub-diagonal consists of all 1's, and all the other elements of the matrix are zeros:

$$J = \begin{pmatrix} \lambda_k & 0 & \dots & 0 & 0 \\ 1 & \lambda_k & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \lambda_k \end{pmatrix}$$

The entries on diagonal are the *eigenvalues* of the matrix. These type of matrices are called simple Jordan matrix. Every square matrix is *similar* to a matrix of the Jordan form.

jump discontinuity A function $f(x)$ is said to have jump disconinuity at some point $x = c$, if both *left-hand* and *right-hand* limits of the function at that point exist, but are not equal:

$$\lim_{x \to c^+} f(x) = a, \quad \lim_{x \to c^-} f(x) = b, \quad a \neq b.$$

The Heaviside function is an example of a function with jump discontinuity at point 0.

K

Kepler's laws of planetary motion:
(1) Every planet revolves around the Sun in an elliptical orbit with the Sun at one focus.
(2) The line joining the Sun and a planet sweeps out equal areas in equal time.
(3) The square of the period of revolution of a planet is proportional to the cube of the length of the *major axis* of its orbit.

kernel of convolution In the convolution integral

$$(K * f)(x) = \int_{-\infty}^{\infty} f(t)K(x-t)dt$$

the function K, if we consider f as a parameter.

kernel of integral transform In the *integral transform*

$$Tf(x) = \int_a^b K(x,t)f(t)dt,$$

the function $K(x, y)$. Transforms vary, depending on the nature of the kernel. We get the most common *Fourier* and *Laplace* transforms, when the kernel is equal to e^{ixy} and e^{-xy} respectively.

kernel of linear transformation Let T be a *linear transformation* from one vector space V to another, W. The set of all vectors $\mathbf{x} \in V$, such that $T\mathbf{x} = 0$ is the kernel of the transformation. The kernel is in fact a *subspace* of V: if $\mathbf{u}, v$ belong to the kernel then for any constants α and β the vector $\alpha\mathbf{u}+\beta\mathbf{v}$ also belongs to the kernel. The dimension of the kernel is called *nullity.*

Kronecker's symbol Also called Kronecker delta. A function of two *discrete* parameters i, j, given by the formula

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}.$$

The parameters i and j are chosen, as a rule, to be integers or whole numbers.

L

Lagrange's identity Consider the *Sturm-Liouville boundary value problem*

$$[p(x)y']' - q(x)y + \lambda r(x)y = 0$$

with boundary conditions

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0,\ \beta_1 y(1) + \beta_2 y'(1) = 0$$

and denote $L[y] = -[p(x)y']' + q(x)y$. Then Lagrange's identity is true:

$$\int_0^1 \{L[u]v - uL[v]\}dx = 0.$$

Lagrange multiplier Assume that the function $f(x,y,z)$ is given in some domain D and is differentiable there. If we want to find the *extreme values* of this function subject to the constraint $g(x,y,z) = k$ (or, otherwise, on the *level surface* $g(x,y,z) = k$), then the *gradients* of both functions f and g are parallel at all extreme points, and hence, there is a real number λ such that $\triangledown f = \lambda \triangledown g$. The constant λ is the Lagrange multiplier. In cases when we wish to find the extreme values of a function subject to two or more constraints, we will have two or more Lagrange multipliers. For the procedure of finding extreme values with the use of Lagrange multipliers see method of Lagrange multipliers.

Laguerre equation The second order differential equation

$$xy'' + (1-x)y' + \lambda y = 0.$$

In the case when $\lambda = m$, a positive integer, the solutions are polynomials, known as Laguerre polynomials. See also *Chebyshev equation*.

lamina A 2 dimensional planar closed surface with mass and density.

Laplace operator For a function $f(x,y,z)$ of three variables, which is two times differentiable by all variables, the expression

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Equivalent notation $\triangledown^2 f$ is also used. See also harmonic functions and *Laplace's equation*.

Laplace transform A special type of *integral transform*, where the *kernel* is the function $K(s,t) = e^{-st}$. Formally, the Laplace transform of a function on the interval $[0,\infty)$ is given by

$$\mathcal{L}f(s) = \int_0^\infty e^{-st} f(t)dt.$$

Laplace transform of a function can exist even if the function itself is not integrable on $[0,\infty)$. In fact, for the existence of the Laplace transform of a function f it is enough that this function be piecewise continuous on any finite interval $[0,N]$ and have a growth no more than exponential function: $|f(t)| \le ke^{at}$ for sufficiently large values of its argument.
Here is a list of Laplace transforms of some common functions, where on the left is the function and on the right is its transform:

$$
\begin{array}{cc}
1 & \dfrac{1}{s}, \quad s > 0 \\[2ex]
e^{at} & \dfrac{1}{s-a}, \quad s > a \\[2ex]
t^p,\ p > -1, & \dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0 \\[2ex]
\sin at, & \dfrac{a}{s^2+a^2}, \quad s > 0 \\[2ex]
\cos at, & \dfrac{s}{s^2+a^2}, \quad s > 0 \\[2ex]
\sinh at, & \dfrac{a}{s^2-a^2}, \quad s > 0 \\[2ex]
\cosh at, & \dfrac{s}{s^2-a^2}, \quad s > 0
\end{array}
$$

One of the most important properties of the Laplace transform (and some other *integral transforms*), is that it allows to substitute certain complicated operations by a simpler ones. For example, the *convolution* of two functions is substituted by multiplication (see corresponding article) and the operation of differentiation could be substituted by the operation of multiplication by the independent variable. More precisely, let $f(t)$ be a piecewise differentiable function on any interval $0 \le t \le A$ and satisfy the inequality $|f(t)| \le Ke^{at}$ for $t \ge M$, where K, a, M are some positive constants. Then the Laplace transform of f' exists for $s > a$ and is given by

$$\mathcal{L}f'(s) = s\mathcal{L}f(s) - f(0).$$

Under similar conditions, we have also a formula for Laplace transform of higher degree derivatives of the function:

$$\mathcal{L}f^{(n)}(s) = s^n \mathcal{L}f(s)$$
$$-s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

These properties are crucial in applications of Laplace transform in solving initial value problems.

Laplace transform method For solving *initial value problems* for *linear non-homogeneous differential equations.* The method is especially important in cases when the right side of the equation is *discontinuous* because the other methods (*undetermined coefficients, variation of parameters, series solutions*) are almost never successful in that case. Example: Solve the equation

$$y'' + 4y = g(t)$$

with the initial conditions $y(0) = 0,\ y'(0) = 0$ and the function $g(t)$ defined to be 0 on the interval $[0, 5)$ and $(t-5)/5$ on $[5, 10)$ and 1 on $[10, \infty)$. This function could be written as

$$g(t) = [u_5(t)(t-5) - u_{10}(t)(t-10)]/5,$$

where the function $u_c(t)$ is defined below in the article *Laplace transforms of special functions.* Applying the Laplace transform to both sides of the equation, using the initial conditions, and denoting $\mathcal{L}y = Y(s)$, we get

$$(s^2+4)Y(s) = \frac{e^{-5s} - e^{-10s}}{5s^2}.$$

Making notation $K(s) = 1/s^2(s^2+4)$ we will get

$$Y(s) = \frac{e^{-5s} - e^{-10s}}{5} K(s).$$

Now, using the *partial fraction* decomposition of the function K we find its *inverse Laplace transform* to be the function $k(t) = t/4 - 1/8 \sin 2t$. Finally, the use of theorems about inversions of Laplace transforms of step functions (see the next article again) gives us the final solution

$$y = \frac{1}{5}[u_5(t)k(t-5) - u_{10}(t)k(t-10)].$$

Laplace transforms of special functions (1) Let $u_c(t)$ be the unit step function (generalization of the *Heaviside function*) defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \ge c \\ 0 & \text{if} \quad x < c \end{cases}.$$

Then

$$\mathcal{L}[u_c(t)] = \frac{e^{-cs}}{s}, \qquad s > 0.$$

In the special case $c = 0$(Heaviside function) the transform is just $1/s$.
(2) If $F(s) = \mathcal{L}[f(t)]$ exists for some $s > a \ge 0$, then

$$\mathcal{L}[u_c(t)f(t-c)] = e^{-cs}F(s), \quad s > a.$$

(3) If the function f is periodic with period T, then

$$\mathcal{L}[f(t)] = \frac{\int_0^T e^{-st}f(t)dt}{1 - e^{-sT}}.$$

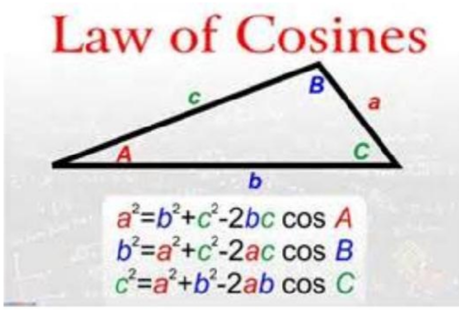
Laplace's equation In partial differential equations. In the simplest case of two variables, the equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0.$$

Solutions to this equation are called *harmonic* functions.

latent roots The same as roots of *characteristic equation* or *eigenvalues.*

Law of Cosines Let α, β, γ denote the (measures of) angles of some triangle ABC and denote by a (the size of) the side, opposite to α, by b– the side opposite to angle β and by c the third side (opposite to γ). Then the following three relationships connect these six quantities:



$$a^2 = b^2 + c^2 - 2bc\cos\alpha,$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta,$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma.$$

Low of Cosines generalizes *Pythagorean theorem*, because when one of the angles is a right angle, then the corresponding cosine is zero and the formula reduces to Pythagorean. This law is used to solve *oblique* (not right) triangles.

Law of large numbers In probability and statistics, this law states that if we chose large number of samples from any given *distribution*, then its *mean* will be close to the mean of the distribution itself. More precisely, as the sample size increases approaching the distribution size (or infinity, if the distribution is continuous), then the sample mean approaches distribution mean.

Law of Sines Let α, β, γ denote the (measures of) angles of some triangle ABC and denote by a (the size of) the side, opposite to α, by b– the side opposite to angle β and by c the third side (opposite to γ). Then the following relations are true:

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c},$$

or, equivalently,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Law of Sines is used to solve *oblique* triangles. For an illustration see *Law of Cosines*.

leading 1's After the process of Gaussian elimination applied to a $m \times n$ matrix, the first non-zero element of each row (except the rows consisting of all zeroes) is 1 and is called leading 1.

leading coefficient For a *polynomial* of degree n,

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

the coefficient of the highest degree term, a_n.

learning curve The same as logistic curve. See *logistic differential equation*.

least common denominator Abbreviated LCD, is the *least common multiple* of two or more denominators of fractions. To find LCD of fractions $\frac{8}{15}$ and $\frac{5}{8}$ we just find the LCM of 15 and 8.

least common multiple (LCM) For given two *natural numbers* n and m, the least common multiple is the smallest whole number that is *divisible* by both of these numbers. Example, for 12 and 22, LCM(12,22)=132. There are methods of finding LCM of two or more numbers, and the most common is described as follows. To find LCM, find *prime factorizations* of that numbers and form a number, that is the product of all prime factors, each of whose is repeated only as many times as the largest multiplicity in any of the numbers. Example: To find LCM of 25, 90 and 120, we write $25 = 5^2,\ 90 = 2\cdot3^2\cdot5,\ 120 = 2^3\cdot3\cdot5$ and LCM(25,90,120)=$2^3\cdot3^2\cdot5^2 = 1800$. There is a formula relating LCM with the greatest common factor (GCF):

$$LCM(n,m) = \frac{n\cdot m}{GCF(n,m)}.$$

least squares regression line Suppose we have gathered data that came in the form of ordered pairs. Then each value geometrically represents a point on the plane. The problem of linear regression is to find a line that represents these values the best. The measure of "closeness" of the line to the points from the data is the sum of the squares of differences between actual values and corresponding values on that line. Among all the possible lines the one that has the smallest sum of that values is called the least squares regression line. As any other line this line has the equation of the form $y = mx+b$, where m is the slope and b is the y-intercept. Traditionally, in Statistics, this equation is written as

$$\hat{y} = b_0 + b_1 x.$$

The notation $\hat{y}$ is used to indicate that these are "predicted", not the actual values from the data. The parameters b_0, b_1 could be found from the data. The slope $b_1 = rs_y/s_x$, where r is the correlation coefficient of the data, s_y is the standard deviation of the second coordinates from the data (y-values) and s_x is the standard deviation of first coordinates from the data (x-values). After the slope is found, the y-intercept b_0 is determined by the formula $b_0 = \bar{y} - b_1\bar{x}$, where the bar on top means the *mean* of the corresponding y or x-values.

least upper bound Also called supremum. If a function f or a sequence a_n are bounded above, then "smallest" of all possible upper bounds is the least upper bound. Example: The sequence $a_n = 1 - 1/n$ is bounded by any number $M \geq 1$ but not by any number less than 1. The number 1 is the least upper bound. See also *greatest lower bound.*

left-hand derivative If a function is not *differentiable* at some point $x = c$, then the limit in the definition of *derivative* does not exists, i.e.

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

does not exist. In some cases, however, one sided limit might exist. If the *left-hand limit*

$$\lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h}$$

exists, then it is called left-hand derivative of f at the point $x = c$. See also *right-hand derivative.*

left-hand limit Let $f(x)$ be a function defined on some interval $[a, b]$ (also could be an *open interval*). Left-hand limit of f at some point c is the limit of $f(x)$ as the point x approaches c from the left, or, which is the same, as $x < c$. The notation is $\lim_{x\to c^-} f(x)$. The precise definition of left-hand limit (the $\varepsilon - \delta$ definition) is the following: The function f has left-hand limit at the point c and that limit is L, if for any real number $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ as soon as $|x - c| < \delta$, $x < c$. See also *limit* and *right-hand limit.*

Legendre equation The differential equation of second order

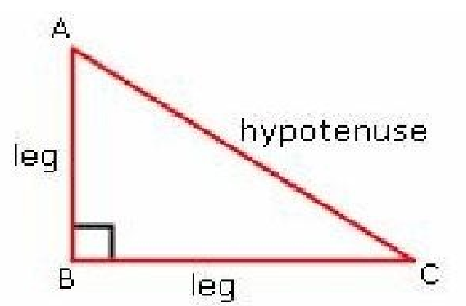
$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0.$$

In case $\alpha = n$, a non-negative integer, the solutions of this equation are polynomials. They are called *Legendre polynomials*, $P_n(x)$ if, additionally, $P_n(1) = 1$.

Legendre polynomials Solutions of the *Legendre equation* for $\alpha = 0, 1, 2, ...$ with the initial condition $P_n(1) = 1$. These polynomials could be expressed in the form

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

legs of a right triangle Two sides of a *right triangle* that form the right angle. The other side is called *hypotenuse.*



length of a line segment If the line segment is on the x-axis and has endpoints a and b, then the length is given by $b - a$. If the line segment connects two points on the plane, then the length is just the distance between these points. See distance formula.

length of parametric curve Let the curve C be given by *parametric* equations $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ and f' and g' are assumed to be continuous on $[a, b]$. Additionally, we assume that when t increases from a to b, the curve C is traversed exactly once. Then the length of the curve C is given by the formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Example: For a circle of radius 1, we have parametric equations

$$x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$$

and

$$L = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t} dt = \int_0^{2\pi} 1\ dt = 2\pi.$$

length of polar curve Let the curve C be given with *polar equation* $r = f(\theta)$, $a \le \theta \le b$. The the length of the curve is given by the formula

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

Example: For the *cardioid* $r = 1 + \sin\theta$

$$L = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta} d\theta = 8.$$

length of a space curve Suppose the curve C is given by a vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \le t \le b$. Then the length of C is given by a formula generalizing the one for the plane:

$$L = \int_a^b |\mathbf{r}'(t)| dt.$$

length of a vector See norm of a vector.

less than An *inequality*, stating that one quantity is smaller than the other, with notation $A < B$. If the quantities are *real numbers*, then this statement means that A is to the left of B on the *number line*. See also *greater than*.

less than or equal to An *inequality*, stating that one quantity is smaller than the other, or equal to it, with notation $A \le B$. If the quantities are *real numbers*, then this statement means that A is to the left of B on the *number line*, or the two points coincide. See also *greater than or equal to*.

level curve For a function $f(x, y)$ of two variables, the equations $f(x, y) = k$ with any real k in the range of f, define the level curves of the function.

level surface For a function $f(x, y, z)$ of three variables, the equations $f(x, y, z) = k$ with any real k in the range of f, define the level curves of the function. Level surfaces of functions of n variables are defined similarly.

L'Hospital's rule Suppose the functions f and g are defined on some interval, containing some point c, with possible exception of that point. Assume also, that the limit of $f(x)/g(x)$ as x approaches c is an indeterminate form $0/0$ or ∞/∞. Then

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$$

implies that also

$$\lim_{x\to c} \frac{f(x)}{g(x)} = L.$$

This rule is the major tool of computation of limits when we have any kind of indeterminate expression. Also, if the application of derivative results in another indeterminate, then the repeated use of this rule still might produce a finite limit. Examples:

(1)

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

(2)

$$\lim_{x\to\infty} \frac{2x^2 - 5}{5x^2 + 3} = \lim_{x\to\infty} \frac{4x}{10x} = \frac{2}{5}.$$

(3)

$$\lim_{x\to 0} x \cot 2x = \lim_{x\to 0} \frac{x\cos 2x}{\sin 2x}$$

$$= \lim_{x\to 0} \frac{\cos 2x - 2x\sin 2x}{2\cos 2x} = \frac{1}{2}.$$

L'Hospital's rule is specifically formulated for two types of indeterminate forms but can be used also for indeterminate forms $0 \cdot \infty$, 0^0, 1^∞ and others. For the first of these cases, if we need to calculate the limit $\lim_{x\to c} f(x)g(x)$ and one of the functions approaches zero and the other one to infinity, then we write the product fg as one of the quotients $\frac{f}{1/g}$ or $\frac{1/f}{g}$ and get indeterminate form $0/0$ or ∞/∞ and proceed as before. For other types the preliminary logarithming the expression results in an indeterminate form of one of the above types and after calculation of that limit we can find the original limit by exponentiating the answer. Example: To calculate the limit

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$$

we logarithm the expression under the limit (denote it by y) we get

$$\ln y = x\ln(1+a/x) = \frac{\ln(1+a/x)}{1/x}.$$

Then, by the l'Hospital's rule,

$$\lim_{x\to\infty}\frac{\ln(1+a/x)}{1/x} = \lim_{x\to\infty}\frac{-a/x^2}{(1+a/x)(-1/x^2)}$$

$$= \lim_{x\to\infty}\frac{a}{1+a/x} = a.$$

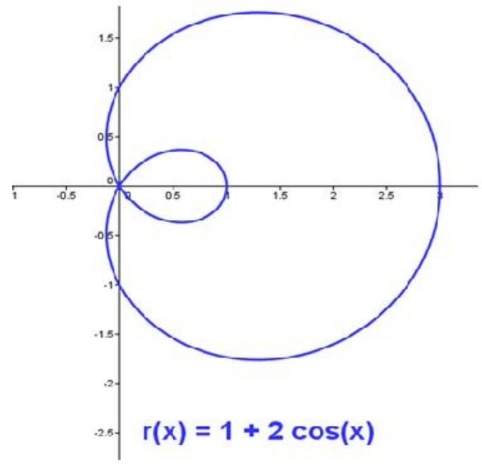
Hence, the limit in question is equal to e^a.

like terms In an *algebraic expression* with one or more variables, the terms where the variable parts are identical (constant factors may be different). For example, in the expression $2x^2y^3 - 3xy^2 + x^3y + 4xy^2$ only the terms $-3xy^2$ and $4xy^2$ are similar and could be combined. The resulting expression is $2x^2y^3 + xy^2 + x^3y$. Also called *similar terms.*

limacon, or more precisely, **limaçon** The family of parametric curves, given in polar coordinates by one of the equations

$$r = a + b\sin\theta, \quad r = a + b\cos\theta.$$

In the spacial case $|a| = |b|$, we have cardioid.



limit One of the most important notions of Calculus. On the intuitive level, the limit of a function at a point or the limit of a sequence when the index grows indefinitely, could be viewed as the eventual value of the function or sequence. In the definitions that follow we will describe precisely these notions for functions and sequences.
(1) Let the function f be defined on some interval I with possible exception of some point c inside of that interval. We say that $f(x)$ has limit L as x approaches c and write

$$\lim_{x\to c} f(x) = L,$$

if for any real number $\varepsilon > 0$ there exists another number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ as soon as $|x - c| < \delta$, $x \neq c$.
(2) There are a few special cases that should be treated separately. First, if the point c is one of the endpoints of the interval I, then the limit should be substituted by one-sided limits. For definitions see left-hand limit, right-hand limit. Second, the case when the point c is the point ∞ or $-\infty$, the definition requires modification. Here is the case $c = \infty$. We say that the function $f(x)$ has a finite limit L as x approaches infinity and write

$$\lim_{x\to\infty} f(x) = L,$$

if for any real number $\varepsilon > 0$ there exists a number $N > 0$ such that $|f(x) - L| < \varepsilon$ as soon as $x > N$. Finally, there is a case when the limit of a function

does not exist in a very specific way of becoming indefinitely large or indefinitely small. In these cases we say that the function has infinite limit, understanding that the limit does not exist as a finite number. One of the possible cases is the following:
We say that $\lim_{x\to c} f(x) = \infty$ if for any real number $M > 0$ there exists a real $\delta > 0$ such that $f(x) > M$ as soon as $|x - c| < \delta,\ x \neq c$.
(3) Limits of sequences could be defined exactly as the limits of functions at infinity, if we view a sequence a_n as a function defined on positive integers only: $a_n = f(n)$. With this view, the exact definition of the limit of a sequence is this:
We say that a numeric sequence $\{a_n\},\ n = 1, 2, 3, ...$ has a limit A and write

$$\lim_{n\to\infty} a_n = A,$$

if for any $\varepsilon > 0$ there exists a positive integer N such that $|a_n - A| < \varepsilon$ as soon as $n \geq N$.

limit comparison test Assume that $\sum a_n$ and $\sum b_n$ are both series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

and $c > 0$ is a finite number, then either both series converge or both diverge.

limit laws Let $f(x)$ and $g(x)$ be functions such that the limits

$$\lim_{x\to c} f(x) \quad \text{and} \quad \lim_{x\to c} g(x)$$

exist. Then the limits of their sum, difference, product and quotient also exist and could be calculated by the rules

$$\lim_{x\to c}[f(x) \pm g(x)] = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x),$$

$$\lim_{x\to c} f(x) \cdot g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x),$$

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

In the last law we assume that $\lim_{x\to c} g(x) \neq 0$.

limit of integration In the notation of the *definite integral* $\int_a^b f(x)dx$ the interval endpoints a and b are called limits of integration or integration limits.

line One of the basic object of Euclidean geometry along with points and planes. Lines are not defined but instead assumed to be understood as having no width and infinite length in both directions. In Cartesian coordinate system lines could be given by equations. The most general equation of the line on the plane is written in the form $ax + by = c$ where $a,\ b,\ c$ are any real numbers. Any line could be uniquely determined by two points (according to one of the Euclidean postulates). If the two points have coordinates (x_1, y_1) and (x_2, y_2) then we define the slope of the line by the relation

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

This quantity does not depend on the choice of the points or the order of that points. Now, the equation of the line passing through that two points could be written using the point-slope form of the line

$$y - y_1 = m(x - x_1),$$

or, equivalently, we could use the coordinates (x_2, y_2).
In the case when the slope of the line and its intersection b with the y-axis (y-intercept) are known, we can use the slope-intercept equation of the line

$$y = mx + b.$$

Each of these forms could be translated into the other, hence, they are equivalent.
The line parallel to x-axis (horizontal line) has zero slope and is given by the equation $y = b$. The slope of the vertical line (parallel to y-axis) is not defined and could be written in the form $x = a$.
Using the notion of the slope we can see that two lines on the plane are parallel if and only if their slopes are equal. If two lines are given by the equations $y = m_1x + b_1$ and $y = m_2x + b_2$, then they are perpendicular if and only if $m_1 \cdot m_2 = -1$.
In three dimensional space any plane is given by the equation $ax+by+cz = d$ and lines could be presented as intersections of two planes in the space. This approach could be extended also to lines in higher dimensional spaces.

For additional information about lines see also *normal line, secant line, tangent line.*

line integral (1) Let C be a curve on the plane given by parametric equations

$$x = x(t) \quad y = y(t) \quad a \le t \le b$$

and let ds denote the element of the length of that curve. Then the line integral along this plane curve could be defined as

$$\int_C f(x, y) ds$$

$$= \int_a^b f(x(t), y(t)) \sqrt{x_t^2(t) + y_t^2(t)} dt.$$

(2) Similarly, if C is a space curve, then we have

$$\int_C f(x, y, z) ds =$$

$$\int_a^b f(x(t), y(t, z(t))) \sqrt{x_t^2(t) + y_t^2(t) + z_t^2(t)} dt.$$

(3) If $\mathbf{r}(t)$ is a *vector function* on the plane or in space, that defines a curve C there, then the line integral along the curve C for a *vector field* $\mathbf{F}$ is defined as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(r(t)) \cdot \mathbf{r}'(t) dt.$$

(4) The fundamental theorem of calculus extends to line integrals of vector functions and has the following formulation:
Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. If f is a differentiable function with continuous *gradient* on C, then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

linear algebra One of the branches of the modern algebra. Deals with *systems of linear equations, matrices, vectors, linear transformations, finite dimensional vector spaces* and many other objects. See corresponding entries for more details.

linear algebraic equation An equation of one or more variables where all the variables are linear. Example: $2x - 3y = 5$. See also systems of linear algebraic equations.

linear approximation Also called tangent line approximation. For a given *differentiable function* f at some point $x = a$, the line that passes through the point $(a, f(a))$ and has the slope $f'(a)$:

$$L(x) = f(a) + f'(a)(x - a).$$

This line provides first degree approximation to the values of the function in small interval around the point a.

linear combination of vectors If $\mathbf{v}_1, v_2, \cdots, v_n$ is a set of vectors in some vector space V and $c_1, c_2, \cdots, c_n$ is a set of *scalars*, then the expression $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ is called linear combination of the set of vectors. For any system of vectors in a linear space this is another vector in V.
Linear combinations of functions and matrices are defined in exactly the same way.

linear dependence and independence of functions Two functions f and g are linearly dependent on some interval I if there exist two constants a and b, not both zero, such that

$$af(x) + bg(x) = 0$$

for all x on I. Otherwise, the functions are linearly independent. This notion could be defined for any number of functions.

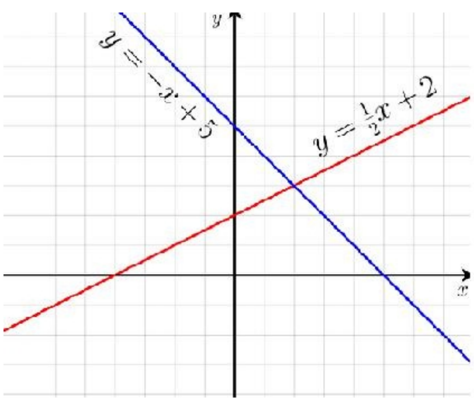
linear dependence and independence of vectors A system of vectors $\mathbf{v}_1, v_2, \cdots, v_n$ is linearly dependent, if there exist *scalars* $c_1, c_2, \cdots, c_n$, not all of which are zero, such that the *linear combination* $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = 0$. If there are no such constants, then the system of vectors is called linearly independent.

linear equation (1) An equation of the form $ax + b = c$, where a, b, c are real (usually rational) numbers. Solution of this equation is $x = (c - b)/a$. Example: $2x - 3 = 6$ has the solution $x = (6 - (-3))/2 = 9/2$.
(2) An equation of two variables of the form $ax + by = c$. This equation has infinitely many

solutions that graphically represent a line on the plane. More generally, an equation with n variables $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is also linear and its solution set represents an $(n-1)$-dimensional *hyperspace*.

linear inequality An expression of the form $ax + b \leq c$, where a, b, c are real numbers, or any other similar expression with relations $<$, $>$ or $\geq$ instead. Linear inequalities with two or more variables are defined exactly like the *linear equations*, except that the equality sign is substituted by one of four inequality signs.

linear function The function, given by the equation $f(x) = ax + b$, where a and b are any *real numbers*. The graph of this function is a line (hence, the term), with the *slope* equal to a and y-intercept b.



linear model If the *mathematical model* of some physical phenomena, process, etc. results in a linear equation(s) (algebraic or differential), them the model is called linear.

linear operator See linear transformation.

linear ordinary differential equations A class of *ordinary differential equations* where the unknown function and all of its derivatives appear in the first order. These equations could be classified further depending on the nature of coefficients (constant or variable) or the number of unknown functions and equations (single equation or system of equations), order of the equation, the presence of the "right side" or its absence (homogeneous or non-homogeneous), etc. The equation $y''' + 2y'' - 4y' + 3y = 0$ is a linear homogeneous equation of the third order with constant coefficients and the equation $y'' + t^2y' - \sin ty = \cos t$ is a linear non-homogeneous equation of the second order with variable coefficients.

The methods of solving homogeneous equations with constant coefficients:

(1) To solve the most general second order equation

$$ay'' + by' + cy = 0 \qquad (1)$$

we assume that the solutions should be exponential functions of the form e^{rt}, where r is some unknown constant to be determined. Substituting in the equation gives

$$(ar^2 + br + c)e^{rt} = 0,$$

or, because $e^{rt} \neq 0$, we have the equation $ar^2 + br + c = 0$ which is just an algebraic quadratic equation, called *characteristic equation*. Depending on the solutions of this equation, we have three possible types of solutions for the differential equation.

(a) The characteristic equation has two real distinct solutions r_1 and r_2. Then the functions e^{r_1t} and e^{r_2t} are the solutions of the equation (1) and the *general solution* of that equation is given by $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$ with arbitrary constants c_1, c_2.

Example: The equation $y'' + y' - 6y = 0$ has characteristic roots $r = 2, -3$ and the general solution of the equation is $y = c_1e^{2t} + c_2e^{-3t}$.

(b) The characteristic equation has on real repeated root r. Then the general solution of the equation (1) is given by the formula $y(t) = c_1e^{rt} + c_2te^{rt}$.

Example: The equation $y'' - 8y' + 16y = 0$ has one repeated characteristic root $r = 4$ and the general solution is $y = c_1e^{4t} + c_2te^{4t}$.

(c) The characteristic equation has two *complex conjugate* roots: $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$. In this case it is still possible to chose real solutions for the equation (1) and the general solution will have the form

$$y(t) = c_1e^{\lambda t}\cos \mu t + c_2e^{\lambda t}\sin \mu t.$$

Example: The equation $y'' + 4y = 0$ has complex conjugate characteristic roots $r = \pm 2i$ and the general solution is $y = c_1 \cos 2t + c_2 \sin 2t$.

(2) Solutions of equations of higher order follow the

same idea as the solution of the second order. To solve the general nth order homogeneous equation with constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

we form the corresponding algebraic characteristic equation $a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$ and find its roots. As in the case of the quadratic equations, we have only three possible outcomes described above as cases (a), (b) and (c). Accordingly, the solutions of the nth order equations are just combinations of the solutions of these three types.
(3) For another approach for solving linear homogeneous equations with constant coefficients see Laplace transform method.

Methods of solutions of non-homogeneous linear equations with constant coefficients are described in entries (articles) undetermined coefficients,variation of parameters, Laplace transform method.
Arbitrary linear equations of the first order (constant or variable coefficients, homogeneous or nonhomogeneous) could be solved by the method of integrating factors.
Equations of order two or higher with variable coefficients are much more difficult to solve. See the entries *Euler equation, series solutions, ordinary points, singular points.*
For solutions of systems of linear equations see the separate entry *systems of differential equations.*

linear programming A branch of applied mathematics dealing with optimization problems: problems of finding the largest or smallest values of a given quantity. Mathematically it is reduced to finding solutions of systems of linear inequalities. Among different methods of solutions the most common are the graphical (geometric) and simplex methods.

linear relationship A relation between two quantities (variables) where one of them depends on the other linearly: If x and y are the variables then $y = ax + b$ where a and b are fixed constants.

linear regression See least squares regression line.

linear speed For an object that moves along some curve. If the speed is constant then it could be determined by the formula $v = s/t$, where s is the distance covered ant t is the time elapsed. For the special case when the curve is a circle of radius r, there is a simple relationship between the linear speed v and *angular speed* ω : $v = r\omega$.

linear systems General term that may refer to either *systems of linear algebraic equations* or *systems of linear differential equations.*

linear term In a *polynomial* the term that contains x. Example: In the polynomial $p(x) = 2x^2 - 3x + 5$ the linear term is $-3x$.

linear transformation Let $T : V \to W$ be a function from one *linear vector space* to another. It is called linear transformation if for any two vectors $\mathbf{u}, \mathbf{v}$ from V and any *scalar* c, the following conditions are satisfied:

$$(a)\ T(\mathbf{u} + v) = T(\mathbf{u}) + T(\mathbf{v})$$

$$(b)\ T(c\mathbf{u}) = cT(\mathbf{u}).$$

In the special case when $V = W$, linear transformations are also called *linear operators.*
The set of all vectors in V that are transformed to the zero vector by the transformation T is called the kernel of T and denoted $ker(T)$. It is a subspace of V in the sense that any *linear combination* of kernel vectors also belongs to the kernel. Also, the set of all vectors in W that are images of at least one vector from V is called the range of T and denoted $R(T)$. The range is a subspace of the image space W. The *dimensions* of kernel and range are called *nullity* and *rank* of T respectively. By one of the important theorems of linear algebra, if the dimension of the space V is n, then $rank(T) + nullity(T) = n$.
A linear transformation $T : V \to W$ is called one-to-one if it maps distinct vectors from V to distinct vectors in W. One-to-one transformations possess the inverse, denoted by T^{-1} and defined by the properties:
If $\mathbf{v} \in V$, then $T^{-1}(T(\mathbf{v})) = \mathbf{v}$, and
If $\mathbf{w} \in W$, then $T(T^{-1}(\mathbf{w})) = \mathbf{w}$.
For additional information see also composition of linear transformations.

linearity The property of being linear. Can refer to any of the situations where linear phenomenon is demonstrated. This includes algebraic equations, differential equations, transformations, and many others.

linearization of a nonlinear equation In differential equations, the procedure of substituting a *nonlinear equation* by a simpler linear equation. For example, the equation of the pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

is nonlinear. To solve this equation we substitute it by the linear equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$$

which is much easier to solve and gives good approximation of the original problem.

linearly independent solutions For differential equations. Two solutions of an equation f and g are linearly independent if from the equality

$$\alpha f(x) + \beta g(x) = 0$$

it follows that $\alpha = \beta = 0$. See also *linear dependence and independence of functions.*

linearly independent sets Usually relates to sets of vectors. See linear dependence and independence of vectors.

local maximum and minimum Let the function $f(x)$ be defined on some interval that contains the point c. The value $f(c)$ is called local maximum of f, if there is some open interval I with center at c such that $f(x) \le f(c)$ for all points x in I. Similarly, the point is local minimum, if the opposite inequality $f(x) \ge f(c)$ is true. For the method of determining local maximums and minimums see first derivative test or second derivative test.

logarithm The mathematical operation that is the *inverse* of the operation of *exponentiation.* As the exponentiation requires a base to rase to a power, logarithms also are meaningful for a base only. According to this approach, $y = b^x$, if and only if $x = \log_b y$. The base b is a positive number, $b \neq 1$. The most common bases are 2, 10, and e. In the case of base 10 the logarithm is called common logarithm and in case e it is called natural logarithm. The following important properties of logarithms are true:
(1) $\log_b(x \cdot y) = \log_b x + \log_b y$
(2) $\log_b(x/y) = \log_b x - \log_b y$
(3) $\log_b x^r = r \log_b x$.
Here $x,\ y > 0$ and r is any real number. See also change of base formula for logarithms and *logarithmic function.*

logarithmic differentiation Let $f(x)$ be a positive function so that the function $\ln f(x)$ is defined. By the chain rule,

$$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)},$$

which is the logarithmic derivative of f. This kind of differentiation is used often then the derivative of the function is more difficult to calculate. Example: If

$$y = \frac{x^2(x-1)^3}{x+1}$$

then $\ln|y| = 2\ln|x| + 3\ln|x-1| - \ln|x+1|$ and

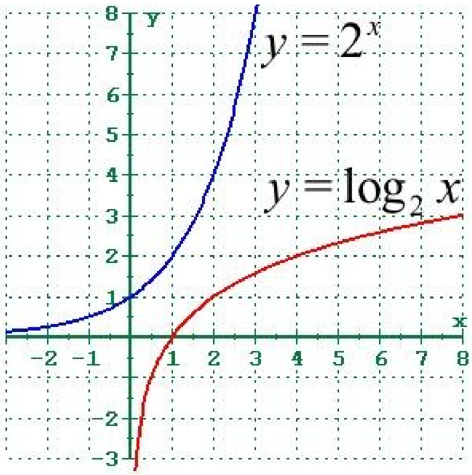
$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{3}{x-1} - \frac{1}{x+1}$$

which results in

$$y' = y\left(\frac{2}{x} + \frac{3}{x-1} - \frac{1}{x+1}\right)$$

$$= \frac{2x(x-1)^2(2x^2+2x-1)}{(x+1)^2}.$$

logarithmic function The function $y = \log_b x$ where $b > 0,\ b \neq 1$. The domain of this function is $(0, \infty)$ and the range is the set of all real numbers. The *inverse* of the *exponential* function. The function $y = \log_e x$ has special notation $y = \ln x$ and in case $b = 10$ the notation $y = \log x$ is used without indication of the base.



logical contrapositive If a logical statement is of the form "If A, then B", (symbolically, $A \Rightarrow B$), then the statement "not $B \Rightarrow$ not A", meaning "If B is not true, then A is not true" is called logical contrapositive. Direct and contrapositive statements are equivalent in the sense that either they both are true or both are false. See also conditional statement.

logistic differential equation The first degree non-linear differential equation of the form

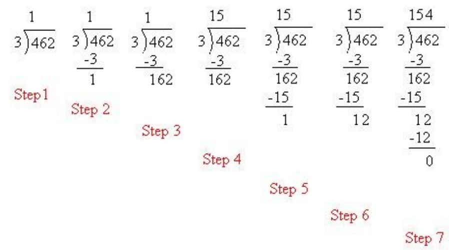
$$y' = r\left(1 - \frac{y}{K}\right)y,$$

where r and K are constants. The solution of this equation with *initial condition* $y(0) = y_0$ is given by

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

and is called logistic curve. The mathematical models described by logistic equation are called logistic models and the function growth is called logistic growth.

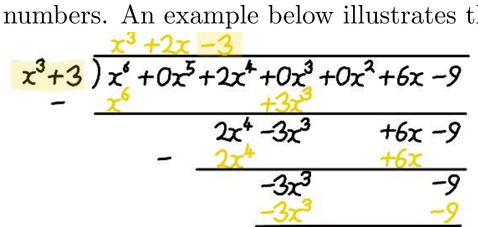
long division of numbers A procedure, algorithm, for dividing two numbers, usually integers or decimals. This method is used in cases when simple division is difficult. For example, to divide 72 by 9 we need only to remember the multiplication table and find the answer to be 8 because we know that $8 \cdot 9 = 72$. On the other hand, if we want to divide 462 by 3, multiplication table is difficult to apply and we use the long division. The procedure is illustrated below.



long division of polynomials When dividing a polynomial $P(x)$ by another polynomial $Q(x)$, where the *degree* of Q is less than or equal to the degree of P, the result is another polynomial plus, as a rule, a remainder. This is expressed as

$$\frac{P(x)}{Q(x)} = D(x) + \frac{R(x)}{Q(x)},$$

where D is the resulting polynomial which degree is always less than the degree of P and R is the remainder polynomial which always has degree less than the degree of Q. The practical procedure of finding the polynomials $D(x)$ and $R(x)$ is called long division (of polynomials) and is similar to the long division of numbers. An example below illustrates this process.



In this example $D(x) = x^3 + 2x - 3$ and the remainder is 0.

When dividing by a binomial of the form $x - c$ the synthetic division is usually simpler and easier to perform. See the corresponding article for details.

lower bound For a function $f(x)$ a number M is a lower bound, if $f(x) \geq M$ for all values of x in the domain of f.

lower triangular matrix A *square matrix* where all the entries above the *main diagonal* are zeros. Example:

$$\begin{pmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ -2 & 5 & 3 \end{pmatrix}.$$

The determinant of such a matrix is just the product of all diagonal elements. See also *upper diagonal matrix.*

lower-upper decomposition of a matrix Or LU-decomposition. If a *square matrix* A can be reduced to *row-echelon* form U by *Gaussian elimination* without interchange of rows, then A can be factored in the form $A = LU$, where L is *lower triangular matrix* and U is an *upper triangular matrix.*

lurking variable In Statistics. Suppose there are two *random variables* related to each other in a way that the change of one of them affects the other. In other words, one of the variables is the *explanatory* (or independent) and the other one is the *response* (or dependent) variables. In the majority of real life situations, however, there are many other aspects (variables) that affect the change of the response variable. Sometimes they are disregarded by some reason intentionally (such as to simplify the situation) and sometimes by omission or a mistake. These type of variables are the lurking variable.

M

MacLaurin series The special case of *Taylor series* for representation of functions as a power series expanded at the point $x = 0$. If the function $f(x)$ is infinitely differentiable (and also *analytic*) in some neighborhood of the point zero, then the power series expansion is true:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Most of the elementary functions have their Maclaurin series expansions in specific intervals:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad (-1, 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad (-\infty, \infty)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad (-1, 1)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad (-\infty, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad (-\infty, \infty)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad (-1, 1).$$

See also binomial series.

magnitude of a vector See norm of a vector.

main diagonal of a matrix For a *square matrix* A, the imaginary line, connecting the element on the first row of the first column with the element on the nth row and nth column. Example: In the matrix

$$A = \begin{pmatrix} 2 & 4 & -2 \\ 0 & -1 & 1 \\ -2 & 5 & 3 \end{pmatrix}$$

the main diagonal goes through the elements 2, -1 and 3.

major axis Assume we have an ellipse with the center at the origin

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with $a > b$. Then the *foci* of the ellipse are located on the x-axis. The line segment connecting the points a and $-a$ on the x-axis is the major axis of the ellipse. In case $a < b$, the major axis is located along the y-axis. The same notion is valid also for translated and rotated ellipses. See also *minor axis*.

mantissa An outdated term, used to indicate the decimal number's fractional part. Was mainly used in logarithmic and trigonometric tables to save space.

map, mapping A term used as an equivalent to the term *transformation* or, sometimes also function.

Markov chain or Markov process. A sequence of events/processes for which the outcome in the next step (or stage) depends on the preceding step. It is also said that in Markov chains the future states depend only on the present state and do not depend on past.

margin of error As a rule, margin of error is defined to be twice the *standard error* of the *sampling distribution*. The interval around the mean of the distribution that has twice the length of the margin of error (one margin of error to the left and one to the right) is the *confidence interval*. Depending on the type of the sampling distribution (of means, of proportions, or something else) the formulas for calculating margins of errors vary significantly. For example, the standard error for proportions is defined as $SE(\hat{p}) = \sqrt{\hat{p}\hat{q}/n}$ and the margin of error is $ME = 2SE(\hat{p})$. Here n is the sample size, $\hat{p}$ is the proportion of "success", and $\hat{q} = 1 - \hat{p}$.

marginal cost function In economics, the term marginal is used as a substitute for derivative with respect to functions under consideration. If $C(x)$ denotes the cost function, then $C'(x)$ is the marginal cost function that shows rate of change of the cost associated with producing x units of some product. The marginal revenue and profit functions are defined in the same way.

mathematical induction Or the method of mathematical induction. One of the main logical tools in proving statements/theorems, especially involving unlimited number of elements. It is actually one of the *axioms* of arithmetic.
The method works as follows: Suppose we have a statement $P(n)$ for each natural number n and the following two conditions are true:
1. $P(1)$ is true.
2. For every natural number k, if $P(k)$ is true then $P(k+1)$ is true.
Then $P(n)$ is true for all natural numbers n.
Example: Prove that for all natural numbers n,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof. Let $P(n)$ be the statement we need to prove. In that case $P(1)$ will mean $1 = \frac{1(1+1)}{2}$ which is true. Now let us assume that $P(k)$ is true:

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

and prove that $P(k+1)$ is true. We have

$$1 + 2 + 3 + \cdots + k + (k+1)$$

$$= (1 + 2 + 3 + \cdots + k) + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

which is exactly the statement $P(k+1)$. This means that $P(n)$ is true.

mathematical model The mathematical (algebraic, analytic, etc.) expression of real life problems or processes. Mathematical models are, as a rule, just approximations of actual processes, because in life and nature there are too many aspects to be included. The criteria of correctness and effectiveness of a mathematical model is its accuracy. If the model results in solutions that (more or less) accurately describe the natural phenomenon then it is considered a good model. Depending on nature of the problem, the model will result in algebraic, trigonometric, differential, or some other type of equations.

matrix Plural: matrices. Formally, a matrix is a collection of numbers, vectors, functions, etc., organized in m rows and n columns and separated by a pair of parentheses, braces, or other convenient separation symbols. A numeric $m \times n$ matrix has the form

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

where the coefficients $\{a_{jk}\}$ are real or complex numbers. The following operations are defined for matrices (more details and examples are provided in corresponding special entries, such as *addition and subtraction of matrices, multiplication of matrices, adjoint of a matrix, inverse of a matrix*, and many others):

(1) The equality of two matrices A and B means that they have the same size (the number of rows and columns are the same for them) and that all elements in the first matrix are equal to the corresponding elements in the other matrix.

(2) Addition (subtraction) of two matrices is possible only if they have the same size. In that case we just add (subtract) elements at identical positions.

(3) Multiplication of a matrix by a *scalar* is defined as multiplication of each element of the matrix by that scalar (constant, number). Division by a scalar is defined similarly.

(4) Multiplication of the matrix A (from the left) by the matrix B (from the right) is possible only if the number of columns in A is equal to the number of rows in B. Hence, if A is of size $m \times k$ and B is of size $k \times n$, then the resulting matrix will have the size $m \times n$. The element of the product in the ith row and jth column (denoted by c_{ij}) is equal to $a_{i1}b_{1j} + a_{i2}b_{j2} + \cdots + a_{ik}b_{kj}$. Generally speaking, multiplication of matrices is not *commutative*. See examples in the entry multiplication of matrices.

(5) For square matrices (the same number of rows and columns) the main diagonal consists of elements located on positions with the same numeric value for rows and columns: $a_{jj}, j = 1, 2, \cdots, n$. The matrix that has all 1's on the main diagonal and zeros elsewhere is the identity matrix of size n. The inverse to a matrix A is a matrix B which multiplied by A (from both left and right) results in identity matrix. If the inverse exists, then it is unique and is denoted by A^{-1}. If the inverse exists, then the matrix is called invertible, otherwise it is called singular or noninvertible.

(6) The transpose of a matrix is a matrix for which the roles of the rows and columns are interchanged. If A is of size $m \times n$, then its transpose is of size $n \times m$. For additional properties and operations with matrices see also *augmented matrix, Gaussian elimination, Gauss-Jordan elimination, adjoint of a matrix, eigenvalues, eigenvectors, similar matrices, zero matrix.*

matrix function A *matrix* where each entry is a function. The matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ sin\theta & \cos\theta \end{pmatrix}$$

describes *rotations* of the plane and is an example of matrix function.

matrix method For solving *systems of linear algebraic equations*. If the system has equal number of equations and unknowns (square system), then it could be written in the matrix form $A\mathbf{x} = \mathbf{b}$, where A is a square matrix of size $n \times n$ and $\mathbf{x}, \mathbf{b}$ are column vectors of size n. Now, if A is *invertible*, then applying its inverse A^{-1} from the left to this equation we will get $\mathbf{x} = A^{-1}\mathbf{b}$ and this is the matrix solution of the system of equations.

matrix transformation A linear transformation $T : V \to W$ of one vector space to another that could be written as

$$T\mathbf{v} = M\mathbf{v},$$

where M is some matrix. If V and W are the *Euclidean spaces* R^n and R^m respectively, then any linear transformation is a matrix transformation.

maximization problems The problem of finding the largest value of some variable quantity. See also *optimization problems.*

maximum and minimum values The largest or smallest values of a function on a given interval. Depending on type of the interval (closed or open) and the function (continuous or not), the function may or may not take that values. Additionally, we distinguish the absolute maximum and minimum and local maximum and minimum values. For the methods of finding these values see corresponding entries.

mean Average of a set of objects (numbers, functions, etc.). See arithmetic mean and geometric mean for definitions.

mean value theorem Let $f(x)$ be a continuous function on some closed interval $[a, b]$ and assume it is differentiable on open interval (a, b). Then there exists (at least one) point $c,\ a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The geometric meaning of this theorem is that there should be at lest one point on the graph of the function where the tangent has the same slope as the line segment connecting two endpoints.

mean value theorem for integrals If the function $f(x)$ is continuous on some interval $[a, b]$, then there exists a point $c,\ a < c < b$, such that

$$\int_a^b f(x)\ dx = f(c)(b - a).$$

measure Measure on a set is a way (a rule) to assign a positive number to each subset of that set (including the set itself). Intuitively, a measure of a set on the real line is its "length", the measure of a plane set is its area, and the measure of a three-dimensional solid is its volume. Exact definition of measure of a set is given in advanced analysis courses.

median For a set of numeric values, the value that is in the middle of that set in the sense that there are exactly equal numbers in the set that are greater than median or less than the median. The median may or may not be an element of the set. Examples:
(1) To find the median for the set of numbers $S = \{2, 7, 3, 9, 6, 2, 4\}$ we first put them in order, repeating the numbers that appear more than once: 2, 2, 3, 4, 6, 7, 9. Now, the number 4 (which belongs to the set S) is the median, because we have three numbers less than 4 and three numbers greater than 4.
(2) For the set $S = \{2, 7, 3, 5, 9, 6, 2, 4\}$ we order it and get 2, 2, 3, 4, 5, 6, 7, 9. Now, because the number of elements is even, there is no number in the set with the property of being in the middle. In this case we take two "middle" ones 4 and 5 and their *arithmetic mean* (4+5)/2=4.5 will be the median of S. In this case median does not belong to S.

method of cylindrical shells One of the methods of evaluating the volumes of three-dimensional solids that are solids of revolution. Suppose we need to find the volume of a solid that is formed by rotating the curve $y = f(x), a \leq x \leq b$, about the axis $x = c$. The method consists in considering small portions of the interval $[a, b]$ with the height $f(x)$ with x being some point in that portion, rotated about the same axis $x = c$. The result is a cylindrical shell. If we calculate the volumes of that shells and add all the small shells' volumes, the result will be the approximate volume of the solid. In the limit, as the sizes of the bases become smaller and the number of shells increase, we will get the exact volume of the solid which is now given by the integral

$$V = 2\pi \int_a^b r(x)f(x)dx,$$

where $r(x)$ is the distance from the point x to the axis of rotation.
Example: Consider the region bounded by the lines $y = x/2, y = 0, x = 2$. Find the volume of the solid that is obtained by rotating this region about the axis $x = 3$.
Solution: The region is a triangle with vertices at the origin and points $(2, 0), (2, 1)$. If we now use the method of cylindrical shells, we will have $r(x) = 3-x$ and

$$V = 2\pi \int_0^2 (3 - x)\frac{x}{2}dx$$

$$= \left(3x^2 - \frac{x^3}{6}\right)|_0^2 = \frac{64\pi}{3}.$$

method of Lagrange multipliers Allows to find the extreme values of a function of several variables using the so-called Lagrange multipliers:
Theorem. To find the maximum and minimum values of the function $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming that these extreme values exist on that surface and $\bigtriangledown g \neq 0$):
(1) Find all the values of x, y, z, λ such that

$$\bigtriangledown f(x, y, z) = \lambda \bigtriangledown g(x, y, z)$$

and

$$g(x, y, z) = k$$

(2) Evaluate the function f at all points (x, y, z) that are found in the previous step. The largest of these values is the maximum value of f and the smallest is the minimum value of f.

method of least squares See least squares regression line.

midpoint formula If (a, b) and (c, d) are two points on the plane, then the point on the middle of line segment connecting these points has the coordinates $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$.

midpoint rule For approximate evaluation of an integral. It the function $f(x)$ is defined on some finite interval $[a, b]$, then we divide that interval into n equal parts of the length $\Delta x = (b - a)/n$, as in the definition of the *Riemann integral*. Then denote the midpoint of the interval $[x_i, x_{i+1}]$ by $\overline{x}_i, 1 \leq i \leq n$. With this notations the midpoint rule states that

$$\int_a^b f(x)dx \approx \Delta x \sum_{i=1}^{n} f(\overline{x}_i).$$

This method is less accurate than the trapezoidal rule or the Simpson's rule. The midpoint rule is easily extended to functions of more than one variables in a very similar way.

minimization problems The problem of finding the smallest value of some variable quantity. See also optimization problems .

minor axis Assume we have an ellipse with the center at the origin

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with $a > b$. Then the *foci* of the ellipse are located on the x-axis. The line segment connecting the points b and $-b$ on the y-axis is the minor axis of the ellipse. In case $a < b$, the minor axis is located along the x-axis. The same notion is valid also for translated and rotated ellipses. See also *major axis*.

minors of matrix Let A be a square matrix with *entries* $\{a_{ij}\}$. The minor of matrix A corresponding to the element a_{ij} is the determinant of the matrix that remains after we erase the ith row and jth column of the matrix A. Example: If

$$A = \begin{pmatrix} -4 & 4 & 1 \\ 0 & -1 & 3 \\ 2 & 5 & 2 \end{pmatrix}$$

then the minor corresponding to the element a_{23} (which is 3) is

$$\det \begin{pmatrix} -4 & 4 \\ 2 & 5 \end{pmatrix} = -20 - 8 = -28.$$

See also *cofactor*.

mixing problems Or mixture problems. A general name for a category of application (word) problems where the objective is to find the amount of two ingredients to be mixed to make a mixture with specified properties. Examples: (1) What quantity of a 60% acid solution must be mixed with a 30% solution to get 6 liters of 50% solution? Solution: Denote by x the amount of the 60% solution and by y the amount of the 30% solution. Then the amount of the acid in x liters will be $0.6x$ liters (because only 60% of it is actually acid) and the amount of acid in the y liters will be $0.3y$ liters. In the 6 liter mixture the amount of the acid will be $0.5 \times 6 = 3$ liters. This results in the equation $0.6x + 0.3y = 3$ and the second relation connecting the two variables is $x + y = 6$ because the total amount is 6 liters. This simple system of linear equations has the solution $x = 4, y = 2$.
(2) A jeweler has a ring weighing 90 g made of an alloy of 10% silver and 90% of gold. He wants to use this ring to make another piece of jewelry with gold content of 75%. How much silver he needs to add? Solution: Denote by x the amount of silver to be added. Then the weight of the new alloy will be

$90 + x$ grams and the amount of gold there will be $0.75(90 + x)$ grams. On the other had this new mixture still contains the same amount of gold as before, which is 90% of 90 grams, equal to 81 grams. Hence, the equation will be $0.75(90 + x) = 81$. The solution is $x = 18$ grams.

mode For a set of numeric values, the number that appears most frequently. A set may have multiple modes or have none at all. Examples: The set $S = \{1, 2, 3, 4, 2, 5, 2, 6\}$ has one mode which is 2. The set $S = \{1, 2, 3, 4, 3, 5, 2, 6\}$ has two modes: 2 and 3, and the set $S = \{1, 2, 3, 4, 5, 6\}$ has no mode, because no numbers appear more than once. In this last case it also could be said that all the elements of the set are modes.

model, mathematical See mathematical model.

modulus of a complex number For a *complex number* $z = x + iy$, the non-negative number $|z| = \sqrt{x^2 + y^2}$. Shows the distance from the point z on the *complex plane* to the origin. Also sometimes called *absolute value* of the complex number.

monomial An algebraic expression that contains constants multiplied by *natural* power(s) of one or more variables. Examples: $2x^2,\ -3x^3y^2,\ xyz$.

monotonic function A function that is either *increasing* or *decreasing*. See corresponding entries.

monotonic sequence A sequence that is either *increasing* or *decreasing*. See corresponding entries.

monotonic sequence theorem If a *monotonic sequence* is bounded, then it has a finite limit.

multiple-angle formulas In trigonometry, formulas relating trigonometric functions of multiple angles with functions of single angle. The most common are the double angle formulas, but triple or quadruple angle formulas also exist. Here is one of possible versions of the triple angle formula for cosine function:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

Other versions of this formula and formulas for other functions also exist but very rarely used.

multiple integral For functions of several real variables. If a function f depends on more than one variable, then the generalization of *Riemann sums* and *Riemann integrals* allows to integrate this function in a domain in n-dimensional *Euclidean space* R^n. The most common cases are the *double integral* and *triple integral*. See corresponding entries for details of definitions.

multiplication of complex numbers (1) Let $z = x + iy$ and $w = u + iv$ be two complex numbers written in the standard form. Then, to multiply these two numbers we multiply all terms with each other and combine the *real* and *imaginary* parts: $z \cdot w = (x + iy)(u + iv) = (xu - yv) + i(xv + yu)$, because $i^2 = -1$. Example:

$$(3 - 2i)(-1 + 4i) = -3 + 2i + 12i - 8i^2$$

$$= -3 + 8 + (2 + 12)i = 5 + 14i.$$

(2) If the complex numbers $z = r(\cos\theta + i\sin\theta)$ and $w = \rho(\cos\phi + \sin\phi)$ are given in trigonometric form, then

$$z\dot{w} = r\rho[\cos(\theta - \phi) + i\sin(\theta - \phi)].$$

multiplication of fractions The product of two fractions is defined to be another fraction with the numerator being the product of two numerators and the denominator the product of denominators. Formally, if $\frac{a}{b}$ and $\frac{c}{d}$ are the given fractions, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

In practice, instead of multiplying directly, we simplify first and then multiply, to avoid possible big numbers. Example:

$$\frac{12}{25} \cdot \frac{15}{32} = \frac{12 \cdot 15}{25 \cdot 32} = \frac{3 \cdot 3}{5 \cdot 8} = \frac{9}{40}.$$

multiplication of functions For two functions $f(x)$ and $g(x)$ their product (result of multiplication) function is defined to be the product of their values: $(fg)(x) = f(x)g(x)$. This function is defined where f and g are both defined.

multiplication of matrices Let A be an $m \times r$ matrix and B an $r \times n$ matrix. Then the product $AB = A \cdot B$ is an $m \times n$ matrix whose entries are determined by the following rule: The entry in row i and column j is the sum of products of all pairs from ith row and jth column. In more detail. Let $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$. Then the product matrix $C = AB$ has entries $c_{ij} = \sum_{k=1}^{r} a_{ik}b_{kj}$. As seen from the definition, matrix product is possible only when the number of columns in the first matrix is equal to the number of rows in the second one. In particular, matrix multiplication is not commutative, because in most cases BA is not even defined when AB is defined. For *square matrices* A, B both products AB and BA are defined but are not equal in general. Examples: (1)Let

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{pmatrix}.$$

(2) If

$$A = \begin{pmatrix} 2 & -3 \\ 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 5 \\ 2 & -1 \end{pmatrix}$$

then

$$AB = \begin{pmatrix} -6 & 16 \\ 8 & 1 \end{pmatrix}, \quad BA = \begin{pmatrix} 5 & 20 \\ 3 & -10 \end{pmatrix}$$

and hence, $AB \neq BA$.

multiplication of polynomials To multiply two polynomials we just multiply all terms of the first one by all terms of the second one and combine all like terms. Formally, if

$$p(x) = \sum_{i=0}^{n} a_i x^i, \quad q(x) = \sum_{j=0}^{m} b_j x^j,$$

then the product is a polynomial of degree $n + m$ given by

$$p(x) \cdot q(x) = \sum_{k=0}^{n+m} c_k x^k,$$

where $c_k = \sum_{i=0}^{k} a_i b_{k-i}$. Example:

$$(x^2 + x - 2)(x^3 - x + 1) = x^5 + x^4 - 3x^3 + 3x - 2.$$

multiplication of power series Assume the functions $f(x)$ and $g(x)$ are represented by *power series*. Then the product of these functions could be represented as a power series (with the interval of convergence being equal to intersection of two intervals of convergence) and that series is the product of the power series of the functions f and g. Example: To multiply the functions e^x and $1/(1 + x^2)$ we write their power series representations and multiply as we would multiply two polynomials:

$$\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)\left(1 - x^2 + x^4 - x^6 + \cdots\right)$$

$$= 1 + x + \frac{x^2}{2} - x^2 - x^3 + \frac{x^3}{6} + x^4 - \frac{x^4}{2} + \frac{x^4}{24} + \cdots$$

$$= 1 + x - \frac{x^2}{2} - \frac{5x^3}{6} + \frac{7x^4}{24} - \cdots.$$

multiplication property of equality Let A, B, C be any algebraic expressions and assume also that $C \neq 0$. Then, if $A = B$, then $A \cdot C = B \cdot C$.

multiplication property for inequalities Let A, B, C be any algebraic expressions and assume also that $C \neq 0$. (1) If $A \leq B$ and $C > 0$, then $A \cdot C \leq B \cdot C$; (2) If $A \leq B$ and $C < 0$, then $A \cdot C \geq B \cdot C$.
Similar statements are true also in the case of other types of inequalities: $<, >, \geq$. Example: $2 < 5$ but $2 \cdot (-3) = -6 > 5 \cdot (-3) = -15$.

multiplication rule for probabilities Let A and B be two independent events. Then the probability that both of these events will happen is given by the formula

$$P(A \cap B) = P(A) \cdot P(B).$$

This formula extends to any number of independent events. For the case when the events are not independent the general multiplication rule for probabilities works. Another notation for multiplication rule is $P(A \text{ and} B) = P(A) \cdot P(B)$.

multiplicative identity An element, that multiplied by any other element of the given set, does not change it. For the sets of real or complex numbers the real number 1 serves as a multiplicative identity. For the set of *square matrices*, the *identity matrix* is the multiplicative identity.

multiplicative inverse An element b of some set is the multiplicative inverse of some other element, a, if $a \cdot b = b \cdot a = 1$, where by 1 the *multiplicative identity* is denoted. For the set of real numbers the inverse of a is denoted by a^{-1} or $1/a$ and it exists for all numbers except $a = 0$. For matrices, the inverse is denoted by A^{-1}.

multiplicity of eigenvalue See eigenvalues of matrix.

multiplicity of a zero Let $p(x)$ be a *polynomial* of degree $n \geq 1$. Then, by the *Fundamental theorem of Algebra* it could be factored as a product of exactly n linear *binomials*: $p(x) = a(x-c_1)(x-c_2)\cdots(x-c_n)$, where $c_1, c_2, ..., c_n$ are some complex numbers and $p(c_k) = 0, k = 1, 2, ...n$. If any of these factors repeat, then the corresponding c_j is called multiple zero of the polynomial. The number of times this factor appears in factorization is the multiplicity of this zero. Example: In polynomial $p(x) = (x-1)(x+1)^2(x-2)^3$ the zeros are $x = 1,\ x = -1,\ x = 2$ and they have multiplicities 1,2 and 3 respectively.

multiplying factor See integrating factor.

mutually exclusive events Two *events* are mutually exclusive, if they cannot happen at the same time. *Complementary events* are always mutually exclusive, but this is also true for other events. When rolling a die, the outcomes 2 and 4 are not complementary but they are mutually exclusive.

N

natural exponential function The *exponential function* with the base equal to e and notation $y = e^x$. In calculus, natural exponential functions allow to simplify many calculations. In particular, $(e^x)' = e^x$ and $\int e^x\, dx = e^x + C$.

natural growth law See exponential decay, exponential growth.

natural logarithmic function The *logarithmic function* with the base equal to e and special notation: $\log_e x = \ln x$. In calculus, natural logarithms allow to simplify many calculations. In particular, we have $(\ln x)' = 1/x$ and

$$\int \ln x\, dx = x \ln x - x + C.$$

natural number The numbers $1, 2, 3, 4 \cdots$. This sequence of numbers goes indefinitely and there is no greatest number in it. Also are called *counting numbers.*

negative angle If the angle is placed in standard position and the terminal side is moved in negative (clockwise) direction, then the angle is considered negative. See also *angle.*

negative definite function A function $F(x, y)$ defined on some domain D containing the origin is negative definite, if $F(0, 0) = 0$ and $F(x, y) < 0$ everywhere else in D. If $F(x, y) \leq 0$, then the function is negative semidefinite. See also *positive definite.*

negative definite matrix A *symmetric* matrix A is negative definite, if for any vector $\mathbf{x} \neq 0$, $\mathbf{x}^T A\mathbf{x} < 0$. If the inequality is substituted by $\leq$, then the matrix is negative semidefinite. Here $\mathbf{x}^T$ indicates the *transpose* of the column vector $\mathbf{x}$. See also *positive definite matrix.*

negative number A number that is less than zero. Equivalently, any number that could be placed on the *number line* to the left of the *origin.*

Newton's law of cooling According to this law, discovered experimentally, the rate of cooling (loss of temperature) of a body is proportional to the difference of temperatures of the body and the surrounding environment (if this difference is relatively small). Mathematically, this law could be expressed by the differential equation

$$\frac{dT(t)}{dt} = -k(T - T_0),$$

where $T(t)$ is the temperature of the body and T_0 is the surrounding temperature. The solution is given by $T(t) = T_0 + (T(0) - T_0)e^{-rt}$.

Newton's laws of motion The laws of classical mechanics established by I.Newton. See also *Newton's second law.*

Newton's method A method for finding *approximate solutions* of some equation $f(x) = 0$, where f is some sufficiently smooth function, not necessarily a polynomial. If the solutions of this equation are not possible by some exact method, then in many cases Newton's method gives good approximations. The procedure is as follows. If we are trying to find the solution of the equation and we have two values a and b, where $f(a)$ and $f(b)$ have opposite signs, we know (by the intermediate value theorem) that there is a solution to the equation $f(x) = 0$ in the interval (a, b). Choose any of the endpoints to be our first approximation to the solution and denote it by x_1. If $f(x_1) \neq 0$, then we choose the second point of approximation using the intersection of the tangent line to f at the point x_1 with the x-axis. This point, which we denote by x_2, will be given by equation $x_2 = x_1 - f(x_1)/f'(x_1)$. Continuing the same way, we have a sequence, given by recurrence formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, 3, ...$$

This procedure does not always give good approximation to the solution. The following statement indicates conditions under which the procedure converges to the solution and also gives estimate of how exact the approximation is.
Theorem. Suppose $f'(x)$ and $f''(x)$ do not change their signs, $|f'(x)| \geq m > 0$ and $|f''(x)| \leq M$ for all values of x in some interval (x_0, x_1). Assume that $f(x_0)$ and $f(x_1)$ have opposite signs, while $f(x_1)$ and $f''(x_1)$ have the same sign. Then there exists a point c in (x_0, x_1) with $f(c) = 0$. Moreover, if $|x_1 - x_0| \leq m/M$, then the successive approximations by Newton's method $x_1, x_2, x_3, \cdots$ approach c and $|x_{n+1} - c| \leq |x_{n+1} - x_n|$.

Newton's second law The force on the object is equal to its mass multiplied by the acceleration of that object.

nilpotent matrix A square matrix A such that $A^n = 0$ for some positive integer $n \geq 1$.

nondegenerate conic section The opposite of *degenerate conic sections.* All the regular conic sections (*ellipses, parabolas, hyperbolas*) are nondegenerate.

nondifferentiable function A function that is not *differentiable.* Among the reasons for a function to be nondifferentiable are: failure to be continuous at the point, having "wedges" at the point. There are continuous functions, that are not differentiable at any point.

nonhomogeneous algebraic equations The system of linear equations where not all right sides are zero. See also *homogeneous algebraic equations.*

nonhomogeneous linear differential equation The equation of the form

$$y'' + p(x)y' + q(x)y = g(x),$$

or a similar one with higher order derivatives. In case when $g(x) = 0$, the equation is *homogeneous.*

noninvertible matrix Also called *singular matrix.* A matrix such that the *inverse* does not exist. See also corresponding entry.

nonlinear differential equation Any differential equation where the unknown function or any of its derivatives appear in a nonlinear form. Examples could be

$$y'' + 2(y')^2 = 0, \quad y'' + y'y = 2.$$

The nonlinear equations could be categorized by the highest degree of the derivative involved, by the existence or non-existence of the function on the right side, and many other criteria, just as in the case of linear equations. The theory of nonlinear equations is not as complete as the theory of linear equations and solving different equations requires involvement of different methods. These include the method of *linearization*, and solutions of *autonomous* equations and systems. For more details see the corresponding entries.

nonsingular matrix A (square) matrix, that has an *inverse*. Opposite of the *singular* matrix.

nontrivial solution Many differential equations have two types of solutions: one is a meaningful function and the other one is a constant or even identically zero solution. These last type solutions are called *trivial* and the first type solutions are nontrivial solutions. Example: The equation $y' = -y^2$ has the trivial solution $y = 0$ and nontrivial solutions $y = 1/(x + C)$ for any real constant C.

norm of a vector If $\mathbf{v}$ is a vector in some vector space V with *coordinates* $(v_1, v_2, \cdots, v_n)$, then its norm is defined to be the non-negative number

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

In the case when V is an inner product vector space, $||\mathbf{v}|| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2}$. Also called the length or magnitude of the vector.

normal The same as *normal vector*.

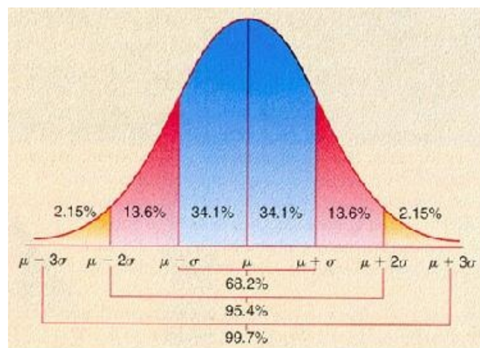
normal density function The function that describes the *normal distribution*. Mathematically it is given by the function of the form

$$f(x) = ce^{(x-a)^2},$$

where c and a are constants specifically chosen to satisfy the conditions of *probability distributions*. For more precise definition see bell shaped curve.

normal derivative The directional derivative in the direction of the *normal vector*.

normal distribution The continuous *probability distribution* expressed by the *normal density function*. The probabilities for this distribution are calculated as areas under the curve. For example, if we want to know what is the probability of choosing a point between the values a and b, then we calculate the area under the curve between a and b. These areas are calculated with the use of tables, graphing calculators or computers. See also standard normal distribution.



normal matrix A matrix A is normal, if $AA^* = A^*A$. Here A^* denotes the conjugate of the transpose of the matrix. In case of real matrices, A^* is just the transpose of A.

normal probability curve The curve that is the graph of the *normal density function*. See the definition and the picture above.

normal system For any *system of linear equations* written in the matrix form $A\mathbf{x} = \mathbf{b}$ the associated equation

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

is called the normal system. Here A^T is the *transpose* of A. The normal system is important in finding the *least square solutions* of the system.

normal vector A vector that is *orthogonal* to a curve or a surface. In case of the plane curve this means that the vector is perpendicular to the tangent line at a given point of the curve. In three dimensional space orthogonality neans that the vector is perpendicular to the tangent plane to the surface at some point of that surface. A line in the direction of the normal vector is called normal line.

normalization To make a system of *orthogonal vectors orthonormal*, the procedure of normalization is necessary. For that we just divide each vector from orthogonal system by its norm and the resulting vectors will have norm 1. See *Gram-Schmidt process* for the details. This term can also be used in other contexts where the notion of "normal" is appropriately defined.

nullity Let T be a linear transformation in some *vector space* V and K denotes the *kernel* of that transformation. The *dimension* of the kernel is called nullity.

null hypothesis In statistics, a hypothesis (statement, assumption) that a certain *population parameter* has some value. This hypothesis is usually denoted by H_0 and has the form *parameter=some value*. For example, if the hypothesis is about the *population mean*, then $H_0 : \mu = \mu_0$. Sometimes the hypothesis is given also in the form $\mu \geq \mu_0$ or $\mu \leq \mu_0$. See also alternative hypothesis, hypothesis testing.

null set The set that contains no elements. Has the same meaning as *empty set* and the same notation $\emptyset$.

nullspace Let A be an $m \times n$ matrix and $\mathbf{x}$ be a vector in R^n. The solution space of the equation $A\mathbf{x} = \mathbf{0}$ is the nullspace of the matrix. The nullspace is the subspace of R^n which is translated to zero by the action of the matrix (linear transformation) A. See also kernel of linear transformation.

number The most basic object in mathematics. The numbers are impossible to define, they rather could be described. We distinguish two major sets of numbers: the real numbers and the complex numbers with the understanding that the first set is a subset of the second one. Additionally, the real numbers consist of subsets of *irrational* and *rational* numbers. Rational numbers further have the subsets of *integers, whole numbers, natural numbers* and *prime numbers*. See the corresponding entries for definitions. The visual presentation of real numbers is done by expressing them as points on the *real line*, and the complex numbers can be expressed as points on the *complex plane*.

number theory The branch of mathematics that deals with the properties of whole numbers. Despite the fact that this is one of the oldest branches of mathematics, there are several unresolved problems, concerning properties of whole numbers. Among them, the famous Riemann hypothesis deals with properties of *prime numbers*.

numerator For a *fraction* or a *rational function*, the top part in the fractional expression. In number $\frac{7}{13}$, 7 is the numerator and 13 is called *denominator*. In function

$$\frac{3x^4 - 2x^2 + x - 5}{x^5 + 2x^2 - 1}$$

the polynomial $3x^4 - 2x^2 + x - 5$ is the numerator.

numerical coefficient Same as *coefficient*.

numerical integration Calculation of *definite integrals* approximately, when exact calculations are impossible or difficult. The same as *approximate integration*.

numerical methods General term for indicating various methods of solving equations when analytic (sometimes algebraic) solutions are either impossible or very complicated. For solutions of algebraic equations see, e.g., *Newton's method*. For numerical solutions of differential equations see *Euler method*.

O

oblique asymptote Also called slant asymptote. The line $y = ax + b$ is an oblique asymptote for the function $f(x)$ if one or both of the relations $\lim_{x\to\infty}[f(x) - (ax + b)] = 0$ or $\lim_{x\to-\infty}[f(x) - (ax + b)] = 0$ hold. This relations mean that for large values of x (positive or negative) the value of the function is approximately the same as the value of the linear function $y = ax + b$. A function may or may not cross its oblique asymptote. Examples: (1) The function $f(x) = \frac{x^2+1}{x+1} = (x - 1) + \frac{2}{x+1}$ has oblique asymptote $y = x - 1$ but it never crosses that line. (2) The function $f(x) = x - 1 + \frac{\sin x}{1+x^2}$ has the same oblique asymptote $y = x - 1$ and they intersect infinitely many times.

oblique triangle A triangle where no two angles and no two sides are equal

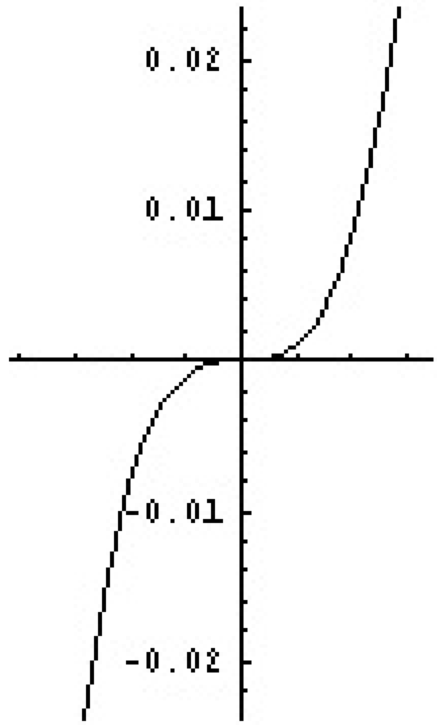
observational study In *statistics*, when the researcher collects data by just observing the objects, without actively trying to affect the outcome. Surveys and polls are examples of observational studies.

obtuse angle An angle that measures between 90° and 180° in *degree measure* or between $\pi/2$ and π in *radian measure.*

obtuse triangle A triangle that contains an obtuse angle.

octant The *Cartesian* coordinate system in the tree-dimensional space creates tree coordinate planes given by the equations $x = 0$, $y = 0$ and $z = 0$ which intersect at the *origin.* These three planes divide the space into eight parts called octants.

odd function A function $f(x)$ of real variable that satisfies the condition $f(-x) = -f(x)$. This condition means that the *graph* of the function is symmetric with respect to the origin. The functions $f(x) = \sin x$ and $f(x) = x^3$ are examples of odd functions.



odd permutations A permutation that is the result of odd number of transpositions. Equivalently, could be defined as a result of odd number of *inversions.*

one-sided limit See left-hand limit and right-hand limit.

one-step methods In approximate (numeric) solutions of differential equations any method that requires the knowledge of values only in one previous step to find values on the next step. The most common example is the Euler method.

one-to-one function Let $f(x)$ be a function defined on some (possibly infinite) interval $[a, b]$. It is one-to-one on the interval, if for any two points $x_1 \neq x_2$ inside that interval $f(x_1) \neq f(x_2)$ and if $f(x) = f(y)$, then $x = y$. In other words, one-to-one functions cannot take the same value more than once. Examples: The functions $f(x) = ax + b$, $f(x) = 2^x$, $f(x) = \ln x$ are all one-to-one. On the other hand, the function $f(x) = x^2$ is not one-to-one, because for any positive number a, $x^2 = a$ gives the solutions $x = \sqrt{a}$ and $x = -\sqrt{a}$. One-to-one functions are important because they possess inverse functions.

one-to-one transformation A *linear transformation* $T : V \to W$ is one-to-one, if it transforms different points of V to different points of W. One-to-one transformations are invertible.

open interval Intervals of the form (a, b) where the endpoints are not included. In this case one or both endpoints may be infinite, so the intervals (a, ∞), $(-\infty, b)$ and $(-\infty, \infty)$ are considered open.

open region A region is called open if for any point P in that region there exists a circle with center at that point that lies completely in the region. The region $D = \{x^2 + y^2 < 1\}$ is open but the region $G = \{x^2 + y^2 \leq 1\}$ is not, because for the latter for points P on the boundary $x^2 + y^2 = 1$ any circle with that center is only partially contained in the region.

operator See linear transformation.

opposite A term that could be used with different meanings in different situations. Most commonly is used as an opposite of a real number a which is $-a$. The opposite of 5 is -5 and the opposite of -2 is $-(-2) = 2$.

optimization problem Any problem where the objective is to find the greatest or smallest possible values of certain variable quantity.

order of a polynomial See degree of a polynomial.

order of differential equation The highest order of derivative present in the given differential equation. In the equation

$$y''' - 2x^2y' + \sin xy = \cos x$$

the order is 3 because y''' is the highest order derivative in that equation.

order of integration In double (or, more generally, multiple) integrals the order in which the repeated (iterated) integration is performed. See *Fubini's theorem* for conditions justifying the change of the order of integration.

order of operations Set of rules that determine the way arithmetic and algebraic operations are performed to avoid ambiguity in calculations. According to this rules the operations are done in the following succession:
1) Parentheses (includes also brackets and braces) from inside out;
2) Exponents;
3) Multiplication and division (left to right);
4) Addition and subtraction (left to right).
Example: $25 - 2^3 \cdot 3 \div (8 + 4) = 25 - 2^3 \cdot 3 \div 12 = 25 - 8 \cdot 3 \div 12 = 25 - 24 \div 12 = 25 - 2 = 23$.

ordered pair Two real numbers a and b grouped together and enclosed in parentheses: (a, b). The order of the numbers is important and (a, b) is different from (b, a). Each ordered pair defines uniquely a point in the *Cartesian* plane and, conversely, each point corresponds to an ordered pair where the numbers are the coordinates of the point. This notion extends to ordered triples also where they now correspond to points in three dimensional space. Similarly, we can generalize it to the case of ordered n-tuples which correspond to points in n-dimensional *Euclidean* space.

ordinary differential equation A *differential equation* that involves only functions and derivatives of functions of one independent variable.

ordinary point For *linear differential equations* with variable coefficients. Suppose we need to find the *series solution* for the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

about the point $x = x_0$. That point is called ordinary point if $P(x_0) \neq 0$. See the entry series solution for details how this fact is used in finding solutions.

ordinate In the plane *Cartesian* coordinate system, the name of the y-axis.

orientation of a curve See positive orientation of a closed curve.

orientation of a surface See positive orientation of a closed surface.

origin In any *coordinate system*, the point from where we start measurements. In particular, in *Cartesian* system origin is the point usually denoted by number 0. In the case of the plane system, for example, the origin is the point of intersection of $x-$ and $y-$axes and has the coordinates $x = 0$ and $y = 0$. Hence, the origin is the point corresponding to the *ordered pair* $(0, 0)$.

orthogonal A term that substitutes the term perpendicular when we deal mainly (but not exclusively) with vectors.

orthogonal basis A basis $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ of a *linear space* V is said to be orthogonal if any pair of vectors in that basis is orthogonal, i.e. $\mathbf{u_i} \cdot \mathbf{u_j} = 0$ for any $1 \leq i, j \leq n, i \neq j$. See also basis.

orthogonal complements Let V be an *inner product* vector space and W its subspace. The set of all the vectors in V that are orthogonal to W is the orthogonal complement of W. The orthogonal complement of any subspace is itself a subspace.

orthogonal curves Two curves are orthogonal at a point where they intersect, if the tangent lines of that curves at that point are perpendicular. Example: The hyperbola $y = 1/x$ and the line $y = x$ are orthogonal at their poit of intersection $(1, 1)$.

orthogonal diagonalization Let A be a square matrix and assume that there exists a *orthogonal matrix* P such that the matrix $P^{-1}AP = P^TAP$ is diagonal. Then A is called orthogonally diagonalizable and the process is called orthgonal diagonalization. It is known that being orthogonally diagonalizable is equivalent of being symmetric.

orthogonal matrices A square $n \times n$ matrix A is orthogonal if its inverse is equal to its transpose: $A^{-1} = A^T$. Equivalently, $AA^T = A^TA = I$, the identity matrix. Orthogonal matrices have many important properties listed below:
(1) The inverse of an orthogonal matrix is orthogonal;
(2) The product of orthogonal matrices is orthogonal;
(3) The *determinant* of an orthogonal matrix is equal 1 or -1;
(4) Both the row vectors and column vectors of the matrix A form an *orthonormal* set in R^n.
Example: The matrix

$$\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$$

is orthogonal.

orthogonal projection Let $\mathbf{v}$ be a vector in R^n and suppose we map that vector onto a line formed by another vector $\mathbf{a}$ not parallel to $\mathbf{v}$. Denote the resulting vector by $proj_{\mathbf{a}}\mathbf{v} = \mathbf{b}$. If this mapping is performed in such a way that the vectors $\mathbf{v}$ and $\mathbf{b}$ are *orthogonal*, then the mapping is called orthogonal projection. The orthogonal projection of a vector on a plane is defined in a similar manner.
Example: The operator of orthogonal projection of any vector (x, y) on the plane onto the x-axis maps that point onto the point $(x, 0)$. The *standard matrix* of that operator is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

orthogonal vectors Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if they form a $90°(\pi/2)$ angle. Equivalently, cosine of the angle between these two vectors is zero. This last condition is also expressed with the use of the *dot product*:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = 0,$$

where $||\mathbf{u}||$ is the *norm (magnitude, length)* of the vector $\mathbf{u}$.

orthogonality of functions Two functions $f(x)$ and $g(x)$ defined on some interval $[a, b]$ (finite or infinite) are called orthogonal, if

$$\int_a^b f(x)g(x)dx = 0.$$

Many systems of functions are orthogonal in this sense. Among them the trigonometric system $\{\cos nx, \sin nx\}$, $n = 0, \pm1, \pm2 \cdots$ (on the interval $[0, 2\pi]$), Chebyshev polynomials (on the interval $[-1, 1]$), Bessel functions, and Legendre polynomials (on appropriate intervals).

orthonormal basis A *basis* $\{\mathbf{u}_1, u_2, \cdots, u_n\}$ of a *linear space* V is said to be orthonormal if any pair of vectors in that basis is orthogonal (in other words, the basis is orthogonal), and each of them has *norm* equal to one: $||\mathbf{u}_j|| = 1, j = 1, 2, \cdots, n$. See also *basis* and *Gram-Schmidt process*.

osculating plane For some smooth space curve let $\mathbf{N}$ and $\mathbf{T}$ denote the *normal* and *tangent* vectors at some point respectively. Then the plane formed by this two vectors is called osculating.

outliers In statistical data all the values that are significantly different (too big or too small) than the most of data are outliers.

overdamped vibration If a particle oscillates and some additional forces, such as friction, affect its movement, then this kind of vibration is *damped vibration* and it's given by the equation

$$my'' + cy' + ky = 0.$$

In case when the *discriminant* of the *characteristic equation* $c^2 - 4mk < 0$, then the vibration is called overdamped. In this case the characteristic roots are negative and the solution has the form

$$y = e^{-(c/2m)t}(c_1 \cos \omega t + c_2 \sin \omega t),$$

and they are quasiperiodic.

overdetermined linear systems A system of *linear equations* where there are more equations than unknown variables. In most cases the overdetermined systems do not have solutions. See also *underdetermined linear systems*.

overlaping sets Two or more *sets* that have common elements. For example, the sets $A = \{x| - 5 < x \leq 2\}$ and $B = \{x|0 \leq x \leq 7\}$ overlap, because the set $\{x|0 \leq x \leq 2\}$ belongs to both of them. The common set is called intersection of two sets and is denoted by $A \cap B$.

P

paired data In statistics has the same meaning as the *ordered pairs* in algebra. The distinction is made because in statistics the paired data comes from collection of *samples* while in algebra the ordered pairs need not have any particular meaning. For the use of paired data see scatterplots.

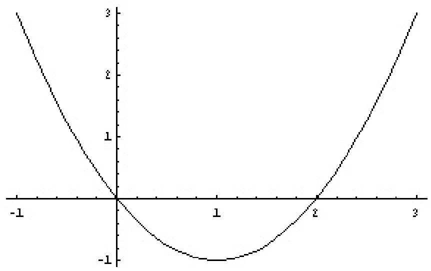
Pappus' theorem Let D be a region of the plane that lies entirely on one side of a line ℓ in that plane. If D is rotated about ℓ, then the volume of the resulting solid is the product of the area A of D and the distance d traveled by the *centroid* of D.

parabola One of the three main *conic sections*. Geometrically, a parabola is the location (locus) of all points in a plane that have the same distance from a given line (called parametrix) and a point (called focus) not on that line. Equivalently, the parabola could be described as the result of cutting a *double cone* by a plane that is parallel to the *generators of the cone*, and not passing through vertex of the cone. Alternative geometric definition could be given with the use of *eccentricity*. See corresponding definition. Algebraically, the general equation of a parabola is given by the quadratic equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where A, B, C, D, E, F are real constants and $A \cdot C = 0$. This means that one of the terms Ax^2 or Cy^2 is missing. The case when both A and C are zero is a case of *degenerate conic*. In the case when the directrix is parallel to the x-axis with the equation $y = -p$ and the focus is located on the y-axis at the point $(0, p)$, the equation of the parabola translates into the *standard form*

$$y = \frac{1}{4p}x^2.$$

Similarly, if the directrix is parallel to the y-axes and the focus is on the x-axis, the equation will be $x = 1/4py^2$. The point where the parabola touches one of the axes is called the vertex. In the more general case when the vertex is shifted, the above equations transform to one of the following:

$$y - k = \frac{1}{4p}(x-h)^2, \quad x - h = \frac{1}{4p}(y-k)^2.$$



All of the above cases happen when in the general equation the term Bxy is missing. In the case $B \neq 0$ the result is still a parabola, which is the result of rotation of one of the previous simpler cases.
The parabola could also be given by its polar equation:

$$r = \frac{d}{1 \pm \cos\theta} \quad \text{or} \quad r = \frac{d}{1 \pm \sin\theta},$$

where $d > 0$.

parabolic cylinder The three dimensional surface given by (for example) the equation

$$a^2x^2 = b^2z.$$

paraboloid See elliptic paraboloid.

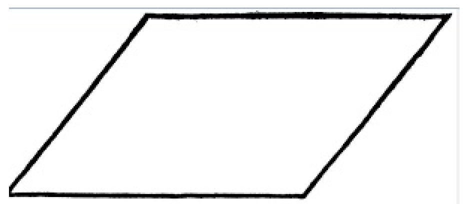
parallelepiped A three dimensional solid, similar to rectangular box. Difference is that if for rectangular solid each face is a rectangle, for parallelepiped each face is a *parallelogram*.

parallel lines Two lines on the plane that do not intersect. If two lines are parallel then they have the same *slope*. Example: The lines $y = -2x + 1$ and $y = -2x - 27$ are parallel.

parallel planes Two planes in the space are parallel if they do not intersect. If $a_1x+b_1y+c_1z = d_1$ and $a_2x+b_2y+c_2z = d_2$ are equations of two planes, then they are parallel if and only if $a_1/a_2 = b_1/b_2 = c_1/c_2$ but are not equal to d_1/d_2.

parallel vectors Two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if there exists a real *scalar* c such that $\mathbf{u} = c\mathbf{v}$. For parallel vector the angle between them is either 0 or $\pi(180°)$.

parallelogram A quadrilateral where two pairs of opposite sides are parallel. In parallelogram the opposite sides are equal. In the particular case when all four sides have the same size, the parallelogram is called a *rhombus*.



Parallelogram.

parallelogram law See addition and subtraction of vectors.

parameter In mathematics parameters can be describes as something in between constants and variables. This means that depending on situation the parameter could either be fixed at some numeric value or start changing its values. In the expression

$$\int_0^\infty e^{-\lambda x} dx$$

λ is a parameter and x is the variable of integration. For other case where the term parameter is used in slightly different meaning see *parametric equations*.

parametric curve See *parametric equation*.

parametric equation Many curves that by different reasons cannot be expressed as a graph of a function, can be expressed by parametric equations. Suppose C is a plane curve and the coordinates (x, y) of the curve are given by the functions $x = f(t)$, $y = g(t)$, where f and g are some functions of the variable t on some interval I. The equations for x and y are called parametric equations of the curve C and t plays the role of parameter in this case.
Examples: (1) The circle given by $x^2 + y^2 = 1$ could be written in parametric form as $x = \cos\theta$, $y = \sin\theta$, $0 \leq \theta \leq 2\pi$.

(2) The parametric equations $x = t^2 - 1,\ y = t/2,\ -2 \le t \le 3$ represent a portion of the parabola opening to the right.

parametric surface In three dimensional space, the variables (x, y, z) on some surface S sometimes could be written in the form $x = f(u, v),\ y = g(u, v),\ z = h(u, v)$, where (u, v) are parameters from some region G. If this is possible then the surface S is called parametric and the above equations are called parametric equations of the surface.

parametrization The process of changing regular equation to parametric equation.

parentheses The symbols ( ). One of the *grouping symbols* along with brackets and braces. Primarily is used to separate certain numbers and variables to indicate operations to be done first.

partial derivative For functions of two or more variables there is no notion of the derivative but instead partial derivatives are considered for each variable. Let $f(x, y)$ be a function of two independent variables and assume that the limit

$$\lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

exists for any fixed y. Then this limit is called the partial derivative of f with respect to the variable x. The notations $f_x(x, y)$ or $\frac{\partial f(x,y)}{\partial x}$ and some others are used. The partial derivative $f_y(x, y)$ is defined similarly. The definitions of partial derivatives are similar for functions of three or more variables.

partial differential equation A differential equation that involves functions of several variables and their partial derivatives. See also heat equation, Laplace's equation, harmonic functions.

partial fraction expansion Any rational function of the form $R(x) = P(x)/Q(x)$, where P and Q are polynomials, could be written in the form of the sum of simpler rational functions, called partial fractions. In order to find this decomposition, as a first step we need to assure that the fraction is proper and if it is not, perform the division and deal with the remainder only, which is necessarily proper. On the next step we factor the denominator into linear and quadratic factors. Depending on types of factors we have different types of partial fraction decompositions.
(1) All the factors are linear and none of them is repeated: $(a_1x+b_1)(a_2x+b_2)\cdots(a_kx+b_k)$. In this case the rational function will have the decomposition

$$R(x) = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k},$$

where $A_1, A_2, \cdots, A_k$ are just constants.
(2) All the factors are linear but some of them are repeated (possibly many times). Suppose the factor $ax+b$ is repeated k, i.e., the denominator has the factor $(ax+b)^k$. Then the decomposition corresponding to this particular factor will have the form

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k}.$$

The partial fractions corresponding to the non-repeated linear factors will be as in the first case.
(3) The denominator has non-reducible quadratic factor ax^2+bx+c. Then the partial fraction corresponding to this particular factor will have the form

$$\frac{Ax + B}{ax^2 + bx + c}.$$

(4) The denominator has a repeated quadratic factor $(ax^2 + bx + c)^k$. Then the corresponding partial fraction will have the form

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}.$$

Examples: 1) To expand the rational function

$$f(x) = \frac{x + 7}{x^2 - x - 6} = \frac{x + 7}{(x - 3)(x + 2)}$$

we write

$$\frac{x + 7}{(x - 3)(x + 2)} = \frac{A}{x - 3} + \frac{B}{x + 2},$$

where A, B are yet to be determined coefficients (this is called the method of *undetermined coefficients*). Multiplying both sides by the common denominator we get $x + 7 = A(x + 2) + B(x - 3)$ and expanding and equating coefficients of similar powers we get a system of two linear equations with unknowns A and B:

$$A + B = 1$$

$$2A - 3B = 7$$

This system has solution $A = 2,\ B = -1$ and we have

$$f(x) = \frac{2}{x-3} + \frac{-1}{x+2}$$

which is the partial fraction decomposition for the function $f(x)$.
2) The function

$$f(x) = \frac{5x^2 + 20x + 6}{x(x+1)^2}$$

should have the expansion

$$f(x) = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Again, multiplying both sides by the common denominator and equating the coefficients we will get a system of three linear equations with unknowns A, B, C. Solving the system we will have $A = 6,\ B = -1,\ C = 9$ and the decomposition

$$f(x) = \frac{6}{x} + \frac{-1}{x+1} + \frac{9}{(x+1)^2}.$$

3) For the function

$$f(x) = \frac{8x^2 + 13x}{(x^2+2)^2}$$

we have the expansion

$$f(x) = \frac{Ax + B}{x^2+2} + \frac{Cx + D}{(x^2+2)^2}$$

and the same method as before gives $A = 8,\ B = 0,\ C = -3,\ D = 0$. Now, the expansion is

$$\frac{8x^2 + 13x}{(x^2+2)^2} = \frac{8x}{x^2+2} + \frac{-3x}{(x^2+2)^2}.$$

See also integration by partial fractions.

partial integration See integration by parts.

partial sum of a series For an infinite series $\sum_{k=1}^{\infty} a_k$ the sum of the first n elements $S_n = \sum_{k=1}^{n} a_k$. Here the terms a_k may be constants or variable terms such as powers of the variable x.

particular solution For *non-homogeneous differential equations* of the type

$$y'' + p(x)y' + q(x)y = g(x)$$

is any solution of this equation. The importance of particular solution is that the *general solution* of this linear equation is the sum of that particular solution and the general solution of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

partitioned matrix A matrix that is divided into smaller parts, usually called blocks. A matrix could be partitioned in many different ways depending on the need.

Pascal's triangle A triangular table of binomial coefficients for easy calculation of binomial expansion. Each entry at the ends of this table is 1, and any other entry is the sum of two entries right above itself.

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}$$

Example: If we want to expand $(x+y)^4$, then we use the numbers on the 5th row and get

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

path Any *curve* could be called a path. Most commonly the term is used to indicate a piecewise-smooth (differentiable) curve.

pendulum equation An equation that describes the movement of a pendulum. Depending on assumptions, the form of the equation may vary. If a pendulum consists of a mass M that is hanging on one end of a weightless rod of the length L and the angle of displacement from the vertical position is denoted by θ, then the movement of the pendulum is described by the equation

$$\frac{d^\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2 \sin\theta = 0. \quad (1)$$

Here $\omega^2 = g/L$, where g is the gravitational constant and the constant γ depends on the damping force acting on the mass m. This equation is called nonlinear damped pendulum equation. For small θ this equation could be linearized by substituting θ instead of $\sin\theta$ and the corresponding equation

$$\frac{d^\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2\theta = 0 \qquad (2)$$

is called linear damped equation. If the damping coefficient γ (and the corresponding term in (1) or (2)) is removed, then equation (1) becomes nonlinear undamped and (2) becomes linear undamped. In this last case the solution of the resulting equation $\theta'' + \omega^2\theta = 0$ is the function

$$\theta(t) = A\cos\omega t + B\sin\omega t$$

and is called simple harmonic motion. Its period is the number $T = 2\pi/\omega$.

percent One hundredth part of a number. The notation $n\%$ means $n/100$. Examples: $27\% = 27/100 = 0.27$, $436.7\% = 436.7/100 = 4367/1000 = 4.367$.

percentile Suppose we have a large set of numeric values organized in increasing order. We can divide this set into hundred parts by putting "almost equal number" of values in each part. If we denote the first point of division by P_1, second point of division by P_2 and so on, then we will get 99 points of division $P_1, P_2, \cdots, P_{99}$. The points between the smallest value and P_1 belong to the first percentile, the points between P_1 and P_2– to the second percentile, and so on, until we get to the 100th percentile which consists of values between P_{99} and the largest value in the set. Obviously, this is possible only if the set has at least 100 values and even for large sets it is impossible to divide the set into 100 exactly equal parts. See also quartiles.

perfect numbers A number is called perfect if it is the sum of all of its positive divisors except the number itself. 6 is perfect because it has divisors 1, 2, 3 and 1+2+3=6. Also 28, 496 and 8128 are perfect numbers. There are only a few perfect numbers known and it is not known if there are finite or infinite perfect numbers.

perimeter Usually, the length of a closed curve. In some cases just means the curve itself. To find the perimeter of some geometric figure means to find the length of the curve that bounds that geometric figure. The perimeter of any *polygon* is calculated by simply adding the lengths of segments that form the polygon. For example, if the polygon is a *quadrilateral*, then the perimeter is the sum of lengths of all four sides. In cases more complicated than a polygon, usually only the use of calculus allows to find the perimeter. See *arc length* for details.

period Let the function f has the property that there exists positive real number m such that

$$f(x + m) = f(x)$$

for all x in the domain of f. The smallest such number m is called the period of the function f and the function itself is called periodic with period m. The functions $\sin x$, $\cos x$ both have period 2π and the functions $\tan x$, $\cot x$ have period π.

period of simple harmonic motion See *pendulum equation.*

periodic function See *period.*

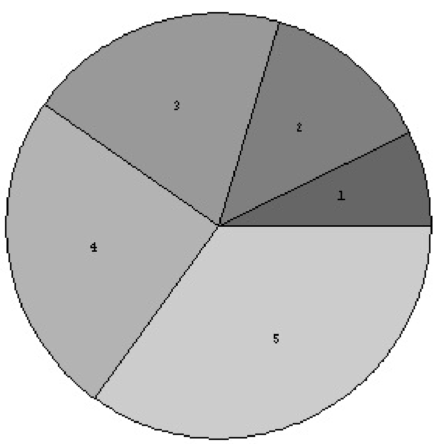
permutation For a given set of integers $\{1, 2, \cdots, n\}$ any rearrangement of the numbers without repeating or omitting any of them, is called a permutation of that set. Any permutation is a combination of finite number of transpositions. transposition is a simplest permutation when the places of only two numbers are interchanged. See also *odd and even permutations.*

perpendicular lines Two lines in the plane are perpendicular if they intersect and form a right (90°) angle. The *slopes* of perpendicular lines are opposite reciprocals: If m_1 and m_2 are the slopes of these lines then $m_1 \cdot m_2 = -1$. Example: The lines $y = 2x - 1$ and $y = -\frac{1}{2}x + 4$ are perpendicular.

perpendicular vectors See orthogonal vectors.

phase shift For trigonometric functions. If the argument of the function $y = \sin t$ is shifted by adding or subtracting a constant, such as in $y = \sin(t - a)$, then this number a is the phase shift of the function. Geometrically this means (for $a > 0$) that the graph of the function is shifted to the right ("delayed") by a units. The same definition is true for all other trigonometric functions.

pie chart One of the ways of visualizing data, usually *qualitative data* along with bar graphs. The picture below shows a data that could be put in five categories and the size of the corresponding shaded area shows the proportion of the values in that category.



piecewise continuous function A function defined on some interval I (finite or infinite) that is continuous everywhere on the interval except finite number of points (in case of bounded interval) or infinitely many points with no limit points inside the interval (in case of unbounded interval). The sawtooth function, and the Heaviside function are examples of piecewise continuous functions.

piecewise defined function A function that is defined differently on different parts of its *domain*. The function

$$f(x) = \begin{cases} x^2 - 1 & \text{if} \quad x \geq 0 \\ 2x + 5 & \text{if} \quad x < 0 \end{cases}$$

is an example of piecewise defined function.

plane One of the basic objects of Euclidean geometry along with points and lines. In *Cartesian coordinate system* planes could be given by equations. The most general equation of the plane is written in the form $ax + by + cz = d$ where $a,\ b,\ c\ d$ are any real numbers. Any plane could be uniquely determined by three points (according to one of the Euclidean postulates). The three coordinate planes ($xy,\ xz,\ yz$) are given by the equations $z = 0,\ y = 0$ and $x = 0$ respectively. Any plane could be determined also by any point on that plane ($P_0(x_0, y_0, x_0$, for example) and the normal vector to that plane. Suppose $P(x, y, z)$ is an arbitrary point on the plane and $\mathbf{r}$ and $\mathbf{r_0}$ are the vectors corresponding to the points P and P_0. Then, if $\mathbf{n}$ is the normal vector to the plane at the point P_0, it is perpendicular to the vector $\mathbf{r} - \mathbf{r_0}$ and, as a result, their *dot product* is zero. That relationship is the determining equation of the plane and is given by the equation $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$. This is called the vector form of the equation of the plane. On the other hand, if the equation of the plane is

$$ax + by + cz = d,$$

then the normal vector $\mathbf{n}$ has the components (a, b, c). The tangent plane to a surface at some point P is a plane that passes through that point and is perpendicular (orthogonal) to the normal vector to the surface at that point. If the equation of the surface is given by the function $z = f(x, y)$, then the equation of the tangent plane to that surface at the point $P_0(x_0, y_0, z_0)$ is given by the equation

$$z - z_0 = \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0).$$

Similar equation is valid also for surfaces given by the equations $F(x, y, z) = k$.

point The most basic geometric object defined to have no measurements. Points on the line, plane, or space could be given by their coordinates. Depending on situation we have one, two, or three coordinates respectively. For example, the point with the coordinates $(1, 2, -3)$ is point in three-dimensional space and the point $(-2, 5)$ is on the xy-plane.

point of inflection See inflection point.

point-slope equation of the line If the *slope* m and a point (x_1, y_1) through which a line passes are known, then the equation of that line is given by the formula

$$y - y_1 = m(x - x_1).$$

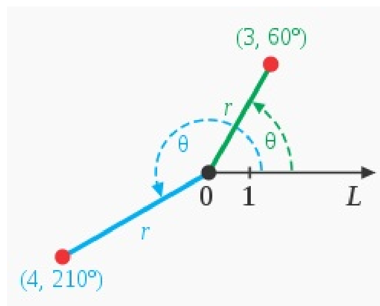
See also *line*.

Poisson distribution A probability distribution describing certain real life situations. If the distribution has the *mean* λ and x denotes the number of "successes", then the probability

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}.$$

The standard deviation of the Poisson distribution is $\sqrt{\lambda}$.

polar coordinates Along with Cartesian coordinate system, the most important method of representing points on the plane. The system consists of a point (the analog of the origin) called the pole and a *ray* coming out from the pole, called polar axis. By convention polar axis is drown horizontally going to the right. To represent a point on the plane in polar coordinates we measure the distance from the point to the pole (usually denoted by r) and measure the angle formed by the polar axis and the line segment connecting the point and the pole (usually denoted by θ). The angle could be measured by either the *radian measure* or *degree measure.* Also the "distance" is allowed to be both positive or negative (see explanations below). This way we put each point into correspondence with a pair of real numbers as in the case of Cartesian system. The major difference is that while to each pair of real number (r, θ) there is exactly one corresponding point, the points themselves could be represented in infinitely many ways. For example, the point $(2, \pi/4)$ could be also written as $(2, \pi/4 + 2\pi n)$ for any integer n because geometrically they represent the same point. Additionally, the same point could be written as $(-2, 5\pi/4)$ because this means that we move the angle $5\pi/4$ radians (and this is opposite to the angle $\pi/4$) and then move 2 units backwards along the side of this angle. Additionally, the same point now could be written as $(-2, 5\pi/4 + 2\pi n)$.
The picture below shows the presentation of two different points with angle measured in degrees.



There are simple relations between rectangular and polar coordinates of a point. If the polar coordinates are known, then rectangular coordinates are found by the formulas

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Conversely, if the rectangular coordinates are known, then

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

This second pair of equations show why the polar representations are not unique. The "radius" r has two possible values $r = \pm\sqrt{x^2 + y^2}$ and the angle θ has infinitely many possible values (differing by an integer multiple of π).

polar equations Equations with respect of the polar coordinates, most generally written in the form $F(r, \theta) = 0$. In practice, however, the equations are usually written in a simpler form $r = f(\theta)$. Many plane curves that are difficult (and sometimes impossible) to write in rectangular system have fairly simple representations in polar coordinates. Additionally, many equations that are not functions in rectangular system turn out to be simple functions in polar coordinates. For example, the equation of the circle centered at the origin with the radius a $x^2+y^2=a^2$ which is not a function, could be written as a simple function $r=a$ in polar coordinates. Here are other examples of curves in rectangular and polar coordinates: Line $y=ax, \quad \theta=a$, circle with center $(1,0)$ and radius 1: $(x-1)^2+y^2=1, \quad r=2\cos\theta$. Most of the polar curves are usually difficult to represent and especially graph in rectangular coordinates. For examples of more polar curves and their graphs see the entries *cardioid, limaçon, four leaf rose* and others.

polar form of a complex number See trigonometric form of a complex number.

polygon A geometric figure made up of three or more segments of a straight line, connected at the endpoints. Polygon with three sides is called a *triangle*, with four sides - a *quadrilateral*, with five sides - a *pentagon*, and so on. See also *regular polygon.*

polynomial The sum of finite number of combinations of *power functions*. Polynomials, as a rule, are written in the decreasing order of powers, but in some cases it is convenient to write them in increasing order. The general form of the polynomial of degree n is

$$p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0.$$

The polynomial $p(x)=5x^4-3x^2+2x+4$ is a polynomial of degree 4.

pooled estimator In statistics, when comparing two proportions, means, variances, etc., often the values from two populations are "pooled" together and the resulting numeric values are called pooled estimators. Example: Suppose we need to compare proportions of two populations with the following information. Sample sizes are are n_1 and n_2 respectively and the numbers of "successes" are m_1 and m_2. Accordingly, the proportions would be $p_1=m_1/n_1$ and $p_2=m_2/n_2$. In this notations, the pooled success proportion would be

$$p_{pooled}=\frac{m_1+m_2}{n_1+n_2}.$$

positive angle An angle in standard position is considered positive if the *terminal side* is achieved by moving counterclockwise starting from the *initial side*. See more details in the article angle.

positive definite form A *quadratic form* is positive definite if it is positive for all values of variables x_j except when all of them are zero:

$$\sum_{i,j=1}^{n} a_{ij}x_ix_j>0$$

unless $x_1=x_2=\cdots=x_n=0$. If the quadratic form is represented by a matrix A then the corresponding matrix is called positive definite. See also *negative definite matrix.*

positive definite function A function $F(x,y)$ defined on some domain D containing the origin is positive definite, if $F(0,0)=0$ and $F(x,y)>0$ everywhere else in D. If $F(x,y)\geq 0$, then the function is positive semidefinite. See also *negative definite function.*

positive definite matrix A *symmetric* matrix A is negative definite, if for any vector $\mathbf{x}\neq 0$, $\mathbf{x}^TA\mathbf{x}>0$. If the inequality is substituted by $\geq$, then the matrix is positive semidefinite. Here $\mathbf{x}^T$ indicates the *transpose* of the column vector $\mathbf{x}$. See also *negative definite matrix.*

positive function A function that takes only positive values.

positive number A number that is greater than zero.

positive orientation of a simple closed curve is agreed to be the counterclockwise direction as we move along the curve.

positive orientation of a closed surface is agreed to be the one for which the *normal vector* points outward from the region enclosed in surface.

power function The function $f(x)=x^n$, where n is a whole number, $n=0,1,2,\cdots$.

power law of limits For any function $f(x)$ and positive integer n,

$$\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a}f(x)\right]^n.$$

In the particular case when $f(x) = x$ we have $\lim_{x\to a} x^n = a^n$.

power series A series formed by the combinations of *power functions*:

$$c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots + c_nx^n + \cdots$$

The numbers $c_0, c_1, c_2, \cdots$ are called the coefficients of the power series. Each power series converges on some interval of the form $-R < x < R$, where R is called the *radius of convergence*. Depending on coefficients, the radius R may be zero (series converges only at $x = 0$), infinite (series converges for any real x), or finite. In this last case the series may or may not converge at the endpoints $x = -R$ or $x = R$ (see also radius of convergence). The power series could be differentiated or integrated *term-by-term* just as any polynomial. In the more general case, also power series centered at and arbitrary point $x = a$ are considered:

$$c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

This series has the exact same properties as the series centered at zero.

prime number A *natural number* that cannot be divided evenly by any other natural numbers except 1 and that number itself. All the other natural numbers are called *composite*. The number 1 is considered neither prime nor composite. First few prime numbers are: 2,3,5,7,11,13,17,19,23,29,31... By the *Fundamental theorem of arithmetic* any natural number is the product of prime numbers.

principal nth root Let a be a real number that has at least one nth root. The principal nth root of a is the one that has the same sign as a and denoted by $\sqrt[n]{a}$. For $a = 4$ the number $\sqrt{4} = 2$ is the principal square root because it is positive. For $a = -27$ the number $\sqrt[3]{-27} = -3$ is the principal cubic root because it is negative.

principle of mathematical induction See mathematical induction.

principle of superposition for linear homogeneous differential equations states, that if y_1 and y_2 are solutions of that equation then so is their sum function $y_1 + y_2$.

probability (1) One of the main branches of mathematics dealing with events which outcome is governed by chance.
(2) Probability of an event is the "likelihood" that that event will happen. It is measured by a numeric value that varies between 0 and 1. The event that cannot happen (an *impossible event*) has probability zero and an event that is certain to happen has probability one. For *discrete variables* the probability of some event A can be calculated by the formula

$$P(A) = \frac{number\ of\ times\ A\ happened}{number\ of\ total\ experiments},$$

where by "experiment" we understand observations we produced to see if A happens. The simplest example is tossing a coin multiple times and observing how many times we get the "tail" (the event A).
If the number of outcomes is some fixed number, such as in the case of tossing a coin (exactly two possible outcomes: "head" or "tail"), or rolling a dye (six possible outcomes: the values from 1 to 6), then the probability of an event could be calculated by the formula

$$P(A) = \frac{number\ of\ ways\ A\ happens}{number\ of\ total\ outcomes}.$$

For operations with probability values see *multiplication rule for probabilities, addition rule for probabilities*. For probability distributions see the entry *distribution* and the related entries mentioned in that article. See also *conditional probability*.

probability model A *mathematical model* where the numeric values used to construct that model are probabilities of some events. For examples see *binomial distribution, Poisson distribution, normal distribution*.

probability density function A function that is determined by the probabilities of some event(s). The defining properties of these functions are:
(1) The function is always non-negative;
(2) The area under that function is exactly one.
See also entries related to specific *probability models*.

product The result of *multiplication* of two (or more) objects. The objects may be real or complex numbers, functions, matrices, vectors, etc. For specific definitions in all of these cases see the entries *multiplication of complex numbers, of fractions, of functions, of matrices* and also *cross product, dot product, scalar product.*

product formulas Also called product-to-sum formulas. In trigonometry, the formulas

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\sin x \cos y = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\cos x \sin y = \frac{1}{2}[\sin(x+y) - \sin(x-y)].$$

Similar formulas are also possible for other functions but hardly ever used. See also *trigonometric identities.*

product rule For differentiation. If the functions f and g are differentiable, then

$$[f(x)g(x)]' = f(x)g'(x) + g(x)f'(x).$$

Examples: (1) $(x^2 \sin x)' = 2x \sin x + x^2 \cos x$,
(2) $(\tan x \ln x)' = \sec^2 x \ln x + 1/x \tan x$.

projection The mapping of a point onto some line or plane. The most common projections are the *orthogonal projections.* See the corresponding entry for the details and examples.

proper value See eigenvalue.

proportion A statement that two or more *ratios* are equal. Ratios may contain just numbers or variables or even functions. Examples of proprtions are $\frac{3}{5} = \frac{6}{10}$, $\frac{x}{3} = \frac{7}{12}$ or the *Law of Sines*:

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{sin \gamma}{c}.$$

To solve a proportion means to find the unknown quantity. For example, to solve the proportion $\frac{x}{3} = \frac{7}{12}$ we cross-multiply and get the equation $12x = 7 \cdot 3$ and $x = 7/4$.

P-value Also sometimes called probability value. One of the methods of hypothesis testing. The P-value is the probability of observing a value like the given value (or even less likely) assuming that the *null hypothesis* is true. The smaller the P-value the more evidence is there to reject the null hypothesis. Big P-values indicate that there is not enough evidence to reject the null hypothesis (so we have to accept it). Visually, the P-value is the area under the *normal probability curve* from the observed vaue to infinity (in the case of right-tailed test), the area from the observed value to minus infinity (left-tailed test), or the sum of areas in both directions (two-tailed test).

Pythagorean identities The trigonometric identities

$$\sin^2 \theta + \cos^2 \theta = 1$$

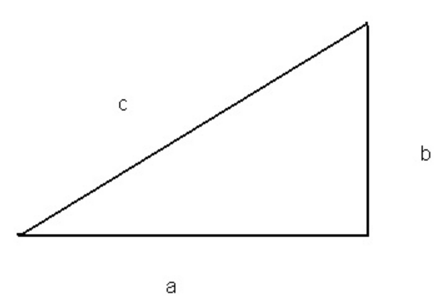
$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta.$$

See also trigonometric identities.

Pythagorean theorem One of the most famous theorems of geometry that states that the square of the length of the *hypothenuse* of any *right triangle* is equal to the sum of the squares of the lengths of the two *legs.* Symbolically, if a, b represent the lengths of legs and c is the length of the hypothenuse, then

$$a^2 + b^2 = c^2.$$



Q

quadrants The $x-$axis divides the *Cartesian plane* into upper and lower half-planes. The addition of the $y-$axis divides the plane again and as a result four quadrants are formed. The quadrants are counted starting with the one corresponding to inequalities $x > 0$ and $y > 0$ and moving against the clock.

quadrantal angle Any angle in standard position where the terminal side coincides with any of the coordinate axes. Hence, quadrantal angles can measure $0°$, $90°$, $180°$, $270°$ or the same sizes plus any multiple of $360°$ in degree measure. In *radian measure* quadrantal angles measure 0, $\pi/2$, π, $3\pi/2$.

quadratic approximation Approximation of a given function by a quadratic function (polynomial). Taylor and Maclaurin polynomials of second degree represent examples of quadratic approximation. For example, for the exponential function $f(x) = e^x$ the Maclaurin polynomial $T_2(x) = 1 + x + \frac{x^2}{2}$ is its quadratic approximation near the origin.

quadratic equation The equation

$$ax^2 + bx + c = 0.$$

This equation always has solution(s) given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The expression under the square root sign is called the *discriminant* and it determines what kind of zeros (solutions, roots) this equation has. (1) The equation $2x^2 - 3x - 4 = 0$ has, by the quadratic formula, two distinct real solutions $x = \frac{3}{4} \pm \frac{\sqrt{41}}{4}$. (2) The equation $x^2 - 2x + 3 = 0$ has two distinct complex roots $x = 1 \pm i\sqrt{2}$ and (3) the equation $x^2 - 2x + 1 = 0$ has two repeated roots $x = 1$. For another method of solutions of quadratic equations see also *factoring*

quadratic forms Quadratic form of n variables $x_1, x_2, \cdots, x_n$ is the expression of the form

$$\sum_{i,j=1}^{n} a_{ij} x_i x_j.$$

For the case of two variables x and y the quadratic form is $ax^2 + by^2 + cxy$.

quadratic formula See *quadratic equation.*

quadratic function Also called quadratic polynomial. The function of the form $f(x) = ax^2 + bx + c$ where a, b, c are numeric *coefficients* (real or complex) and x is the variable. The *graph* of the quadratic function is a *parabola.* Every quadratic function could be written in the standard form $f(x) = a(x - h)^2 + k$, where the point (h, k) is the coordinate location of the vertex of the parabola.

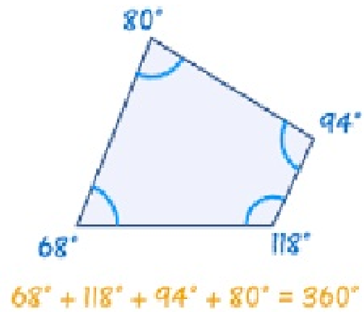
quadratic inequality An *inequality* that involves a *quadratic function.* To solve quadratic inequality means to find all the values of the variable x that substituted into the inequality result in a true statement. Solutions of quadratic inequalities are given either in the form of a finite interval (open or closed) or in the form of the union of two semi-infinite intervals. Examples: The inequality $x^2 - x - 6 > 0$ has the solution set $(-\infty, -2) \cup (3, \infty)$. The inequality $2x^2 - 2x - 4 \leq 0$ has the solution set $[-1, 2]$. Some inequalities have no solution (have empty solution set), such as the inequality $x^2 + 1 < 0$, because the quadratic expression $x^2 + 1$ is always positive.

quadratic surface A three dimensional surface given by a quadratic polynomial which consists of a *quadratic form* plus some linear and constant terms such as

$$x^2 - 2y^2 + z^2 + xy - xz + 3yz - x - 5y + 2z = 8.$$

See also *ellipsoid, hyperboloid, paraboloid.*

quadrilateral Also called quadrangle. A geometric figure that consists of four connected interval of a straight line. Among the most common quadrilaterals are the *square, rectangle, parallelogram, rhombus, trapezoid.*



qualitative data Also called *categorical data*. Data that is organized by its qualitative (as opposed to quantitative) properties. This kind of data may give information in the non-numeric form, such as eye color, agree-disagree opinions, month of the year, etc. It also may contain numeric information put into categories, such as year of birth, ZIP codes and others, that have no particular significance of order.

qualitative variable A variable that takes qualitative values only.

quantitative data Numeric data with added condition that the order of numbers has significant importance. The numeric data of heights is quantitative because people of different heights can be put in specific order (by increasing or decreasing heights). The numeric data of birthdates is not necessarily quantitative because we cannot organize people in order of birthdates (the year of birth is also necessary to do so).

quantitative variable A variable that takes quantitative (numeric) values only.

quartiles Suppose we have a set of numeric values organized in increasing order. We first find the *median* of this set (see corresponding definition for the procedure). On the second step we additionally find the medians of the two halves. This way the set of values is divided into four equal (or almost equal) parts. Each of them is called a quartile. The notations for the three points dividing the set are Q_1, Q_2, Q_3. Note also that second point Q_2 coincides with the median and usually is denoted by M. Finding the quartiles is important for constructing boxplots.

quartic equation Algebraic equation of the fourth degree that could be written in the form

$$\alpha y^4 + \beta y^3 + \gamma y^2 + \delta y + \epsilon = 0, \quad \alpha \neq 0.$$

There is a formula for solving these types of equations which is extremely complicated and difficult to use in practice unlike the *quadratic formula* (see also *cubic equation*). Here is a quick outline of the solution process. First, we divide the equation by $\alpha \neq 0$ and get a *monic equation*

$$y^4 + ay^3 + by^2 + cy + d = 0,$$

then by the substitution $y = x - a/4$ bring it to a simpler form

$$x^4 + px^2 + qx + r = 0, \qquad (1)$$

where the new coefficients p, q, r are found from the old coefficients a, b, c, d. Now we add the expression $2zx^2 + z^2$ to both sides of this equation and bringing some terms to the right side, get

$$x^4 + 2zx^2 + z^2 = (2z - p)x^2 - qx + (z^2 - r),$$

where z is yet to be determined. The left side is the perfect square $(x^2 + z)^2$ and the right side could be made perfect square if we chose z to satisfy the condition $2\sqrt{2z - p}\sqrt{z^2 - r} = -q$. Squaring this relation gives $(2z-p)(z^2-r) = q^2/4$ which is a cubic equation with respect to the unknown z:

$$z^3 - \frac{p}{2}z^2 - rz + \frac{pr}{2} - \frac{q^2}{8} = 0.$$

This equation could be solved by the method described in the article cubic equations and the four solutions of the equation (1) are given by the formulas

$$x_{1,2} = \frac{1}{2}\sqrt{2z - p} \pm \sqrt{-\frac{1}{2}z - \frac{1}{4}p + \sqrt{z^2 - r}},$$

$$x_{3,4} = \frac{1}{2}\sqrt{2z - p} \pm \sqrt{-\frac{1}{2}z - \frac{1}{4}p - \sqrt{z^2 - r}}.$$

For equations where some more terms are missing the solution formula becomes slightly easier. In practice, approximate methods are more appropriate and easier to use than the formulas above.

quasi frequency Some functions are not periodic but have certain repeating pattern. Example: The function $f(x) = e^{-x}\sin 2x$ is not periodic but it oscillates with specific frequency. In this case it is two. The number 2 is the quasi frequency of the function f. The number $2\pi/2 = \pi$ in this case will be the quasi period of the same function.

quotient of numbers The same as the ratio of two numbers. The result of the division of two numbers. $\frac{5}{7}$ is the quotient of numbers 5 and 7.

quotient rule For differentiation of the quotient of two functions. If f and g are two differentiable functions on some interval I, then

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}.$$

See also *differentiation rules.*

R

radian measure One of the two main units used to measure angles, especially useful for applications in calculus. One radian, by definition, is the *central angle* that is formed when a segment equal to the radius of the circle is placed on the circumference of the circle. If we place a segment of the length s on the circumference, the radian measure of the corresponding central angle will be equal $\theta = s/r$, where r is the length of the radius. Since the length of the circle is $2\pi r$, the radian measure of a whole circle will be 2π. See also degree measure.

radicals Also called *roots.* The second meaning of the term is just the sign $\sqrt[n]{a}$ indicating the n-th root of the number a. See the entry roots for all the details.

radical equation Equation that contains the unknown variable under radical of any order. Examples could be $\sqrt{2x+7} - x = 2$ or $\sqrt[3]{3x-1} = 2$. The standard method of solving most of the radical equations is to isolate the radical expression and then raise both sides of the equation to the power of the radical. When solving radical equations containing even order radicals (such as square roots or forth degree roots) this method may produce *extraneous roots.* To avoid getting incorrect roots all the solutions should be checked back into the original equation to assure the validity of that solutions. For example, to solve the equation in our first example, we move the variable x to the right side to isolate the radical first:

$$\sqrt{2x+7} = x + 2$$

and then squaring both sides get the algebraic equation $2x + 7 = x^2 + 4x + 4$. This quadratic equation has two zeros $x = 1,\ x = -3$. After checking in the original equation both of these solutions we see that $x = -3$ is an extraneous solution and the only valid solution of the equation is $x = 1$.
To solve the second example we just raise both sides to the third power (cube both sides) and get a simple linear equation $3x - 1 = 8$ with a single solution

$x = 3$. Extraneous solutions do not arise in this case or in any case where only odd order radicals are involved, so $x = 3$ is a valid solution.

Solutions of some radical equations require repeated use of the method described above. For example, to solve the equation

$$\sqrt{2x-5} - \sqrt{x-3} = 1$$

we first move the second radical to the right and square both sides to get

$$2x - 5 = x - 3 + 2\sqrt{x-3} + 1$$

which still contains a radical. Now we can isolate it, $x - 3 = 2\sqrt{x-3}$ and squaring the second time get the quadratic equation $x^2 - 10x + 21 = 0$ with two solutions $x = 3$, $x = 7$. This time both solutions turn to be valid.

radicand A number or an expression appearing under the *radical.* In the expressions $\sqrt[5]{7}$ and $\sqrt{4x^5 - 3x^3 + x^2 - 1}$, 7 and $4x^5 - 3x^3 + x^2 - 1$ are the radicands respectively.

radiocarbon dating Or just carbon dating. A method of determining the age of old objects using the properties of radioactive carbon C^{14} and the knowledge of its half-life period.

Suppose we need to determined the age of some ancient artifact that has remaining amount of carbon-14 at the amount A. The original amount of the carbon is always the same (in percentages) and is denoted by A_0. The age of the artifact can now be determined by the formula

$$T = -8267 \ln(A/A_0).$$

radius of a circle By definition, a circle is the geometric place (the locus) of all points that have equal distance from a given fixed point. This distance is the radius of the circle. In the general equation of the circle

$$(x-a)^2 + (y-b)^2 = r^2$$

the points (x, y) represent an arbitrary point of the circle, the point (a, b) is the center, and r is the radius.

radius of convergence For a given *power series*

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

there are three possibilities for convergence:
(1) The series converges only for $x = a$;
(2) The series converges for all real x;
(3) There is finite number $R > 0$ such that the series converges for all $|x - a| < R$ and diverges for all $|x - a| > R$.

In the third case we say that the series has finite radius of convergence equal to R. The interval $-R+a < x < R + a$ is called the interval of convergence. At the endpoints $x - a = \pm R$ of the interval of convergence the series may or may not converge. In the first and second cases we say that the radius of convergence is zero or is infinite respectively. Examples: (1) For the series $\sum_{n=0}^{\infty} n!x^n$, $R = 0$; (2) For the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $R = \infty$; (3) For the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1}$$

$R = 1$. This series converges at the right endpoint of the interval $x = 1$ and diverges at the left endpoint of the interval $x = -1$.

random digit A number (usually whole number between 0 and 9) that is chosen as a result of the chance. Tables of random digits are sometimes used in the process of randomization of statistical experiments.

random number The same as random digit with the difference that chosen numbers are not necessarily digits.

random variable A variable that assumes its values randomly (by chance). One of the main objects of *Statistics.*

range of a function The set of all values that the given function $y = f(x)$ can take. Examples: For the function $f(x) = 2x^5 - 3x^2 + 5x - 1$ the range is the set of all real numbers; for the function $f(x) = 2^x$ the range is the set of all positive numbers.

range of a transformation For a linear transformation $T: V \to W$ the set of all vectors $\mathbf{w} \in W$ such that there is at least one vector $\mathbf{v} \in V$ that $T\mathbf{v} = \mathbf{w}$. These values in the range are also called images of the transformation T.

rank (1) For the linear transformation $T: V \to W$ the rank is the dimension of the *range* of that transformation.
(2) For a matrix A the *row space* and *column space* have the same dimension. This dimension is called the rank of the matrix.

rate of change The measure of change of some variable quantity in a unit amount of time. Depending of the nature of change the rate might be a constant or a variable amount itself. For objects moving with constant speed the rate of change is just their velocity. For objects that move at a variable speed the average rate of change in a given time interval is defined to be the difference of their positions divided by the time elapsed: $v_{av} = (s_2 - s_1)/(t_2 - t_1)$. For these type of moving objects the instantaneous rate of change is the limit of the above expression as the time interval approaches zero:

$$v = \lim_{t_2 \to t_1} \frac{s_2 - s_2}{t_2 - t_1}.$$

This last limit is just the derivative of the position function $s = s(t)$.

rate of growth (decline) In differential equation $dy/dt = ry$ the constant r is sometimes called rate of growth or decline depending on its sign (positive or negative).

ratio The result of the division of two numbers or expressions. Ratio of two numbers is more general that a *rational number*, because we can divide any two numbers, not just integers. Hence, $3/4$ is the ratio of the numbers 3 and 4 and also is a rational number, but $\sqrt{3}/4$ is the ratio of the numbers $\sqrt{3}$ and 4 which is not rational (it is *irrational*). The expression

$$\frac{2x^2 - 3x + 1}{x^3 - 4}$$

is the ratio of two algebraic expressions (*polynomials*).

rational equation An *algebraic equation* involving just rational functions. The general form of these type of equations is $P(x)/Q(x) = 0$ where both P and Q are polynomials. The standard method of solving these kind of equations is to multiply both sides of the equation by the denominator function $Q(x)$ and get a polynomial equation $P(x) = 0$. The only difference compared to polynomial equations is that after finding the solutions of the equation $P(x) = 0$ we need to make sure that there are no *extraneous zeros* (roots). To do that we need to plug in into the original equation all the solutions and see if all of them really satisfy that equation. Examples:
1) Solve the equation

$$\frac{2}{x} = \frac{3}{x-2} - 1.$$

Multiplying by the common denominator $x(x-2)$ we get the quadratic equation $2(x-2) = 3x - x(x-2)$ or, after simplification, $x^2 - 3x - 4 = 0$ which has two solutions $x = 4,\ x = -1$. Substituting into the original equation we see that both are solutions of the equation.
2) When we solve the equation

$$2 + \frac{5}{x-4} = \frac{x+1}{x-4}$$

by the exact same method, we get a linear equation with the solution $x = 4$. Now, if we put this solution back into the original equation then it will become undefined because both denominators become zero. As a result we conclude that the equation has no solution.

rational function A function that could be represented as a ratio of two polynomials. The *domain* of a rational function is the set of all real numbers where the denominator does not equal zero. The functions

$$\frac{2x^3 - 3x^2 + 1}{x^2 - 4}, \quad \frac{x-1}{x^2+1}$$

are rational. The first one has the domain $\{x | x \neq \pm 2\}$ and for the second one the domain is the set of all real numbers, because denominator $x^2 + 1$ is never equal to zero.

rational number A *real number* that could be represented as a ratio of two *integers*: $r = n/m$, where $m \neq 0$. Rational numbers can be represented also as *decimals*. The decimal representation of rational numbers is either a terminating decimal or non-terminating, repeating decimal. Examples: $\frac{2}{5}$, $\frac{137}{-218}$, -2.759, $17.247247247....$

rational root test A method of finding roots (zeros, solutions) of a polynomial equation of order 3 or higher. This method does not give a guarantee that these roots will be found, just allows to find them in cases the equation has rational roots.

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial of degree $n \geq 1$ with rational coefficients.Then all the rational roots of the equation $f(x) = 0$ (if they exist) have the form p/q, where p is a factor of the constant term a_0 and q is a factor of the leading term a_n.

Examples: For the equation $2x^3+3x^2-8x+3 = 0$ the possible rational roots are $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$. Checking these values we see that $x = 1, 1/2, -3$ are the roots of the equation. For the equation $x^4-4 = 0$ the rational root test shows that the only possible rational roots are $\pm 1, \pm 2, \pm 4$. The check of all these values shows that none of them are roots of the equation. Instead, the simple factoring $x^4 - 4 = (x^2 + 2)(x^2 - 2) = 0$ shows that the equation has four roots $x = \pm\sqrt{2}$ and $x = \pm i\sqrt{2}$. First two are irrational and the second two are complex. The equation has no rational roots.

ratio test A method of determining the absolute convergence or divergence of a (numeric or functional) series.

(a) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = a < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = a > 1$, then the series is divergent.

(c) In the case $a = 1$ the test cannot give a definite answer.

rationalizing denominators In some algebraic expressions the denominator contains *radical* expressions. The process of getting rid of that irrationality is called rationalizing the denominator.

Examples: (1) In the expression

$$\frac{2}{\sqrt{5x}}$$

we can multiply both numerator and denominator by $\sqrt{5x}$ and get (assuming $x \geq 0$)

$$\frac{2}{\sqrt{5x}} = \frac{2\sqrt{5x}}{\sqrt{25x^2}} = \frac{2\sqrt{5x}}{5x}$$

which has no radical in denominator.

(2)

$$\sqrt[3]{\frac{3x}{2y^2z}} = \sqrt[3]{\frac{4yz^2 \cdot 3x}{8y^3z^3}} = \frac{\sqrt[3]{12xyz^2}}{2yz}.$$

(3) In situations when the denominator contains the sum or difference of two square roots of the form $\sqrt{a}+\sqrt{b}$ or $\sqrt{a}-\sqrt{b}$, the standard method of rationalizing is to multiply both numerator and denominator by the *conjugate* of the denominator:

$$\frac{1}{\sqrt{7}-\sqrt{5}} = \frac{\sqrt{7}+\sqrt{5}}{(\sqrt{7}-\sqrt{5})(\sqrt{7}+\sqrt{5})}$$

$$= \frac{\sqrt{7}+\sqrt{5}}{7-5} = \frac{\sqrt{7}+\sqrt{5}}{2}.$$

ray Take a line and consider only the part that starts at some (arbitrary) point and goes in one direction. The result is a ray.

real axis The same as x-axis. The term is mainly used when considering representation of complex numbers as points on the plane. In that situation the plane will be called *complex plane* and the y-axis will be called *imaginary axis*.

real line The same as *real axis* or *x-axis*. The only difference is that we speak of real line mainly in cases when we want to represent real numbers as opposed to representing complex numbers or ordered pairs. Each real number corresponds to some point on the real line and any point on the real line represents some real number.

real number The set of all real numbers is the union of all *rational* and irrational numbers. In turn, these subsets have their own subsets. The subsets of rational numbers are *prime, natural, whole, integer numbers* and the subsets of irrational numbers are *algebraic* and *trascedental* numbers. Each real number could be represented as a point on the *real line* and each point on the real line represents a real number. These last statements say, in particular, that inside the real numbers there are no "gaps".

real part of a complex number In the *complex number* $z = x + iy$ the number x, which is real. Example: In the complex number $-2 + 3i$ the real part is -2. The number 3 here will be the *imaginary part* of that complex number.

rearrangement of series Any change in order of terms of a series is called rearrangement. Generally speaking, for infinite series, rearrangement changes the sum of the series. The following are two important statements about rearrangements of the numeric series $\sum_{n=0}^{\infty} a_n$.
(1) If the series is absolutely convergent, then any rearrangement of the series leaves the sum unchanged.
(2) If the series is conditionally convergent and L is any real number, then it is possible to rearrange the series in such a way that its sum becomes equal to L.

reciprocal function Most commonly used to indicate the function $f(x) = 1/x$.

reciprocal number Or reciprocal of a number. For a given number $a, a \neq 0$ the reciprocal is the number that is multiplied by a gives 1. Hence the reciprocal of a will be $\frac{1}{a}$. Examples: The reciprocal of -2 is $-1/2$ and the reciprocal of $3/7$ is $7/3$.

reciprocal trigonometric identities The identities

$$\csc\theta = \frac{1}{\sin\theta},\ \sec\theta = \frac{1}{\cos\theta},\ \cot\theta = \frac{1}{\tan\theta}$$

that immediately follow from the definitions of the trigonometric functions.

rectangular coordinate system The same as Cartesian coordinate system.

recurrence formulas, relations Different relations/formulas between the members (terms) of a sequence with the following common feature: To find any term of the sequence (starting from a certain point) it is necessary to know the previous term or terms. Examples: The sequence $\{a_n\}$ given by the formula $a_0 = 2, a_n = 2 \cdot a_{n-1}$ for $n \geq 1$ requires the knowledge of only the previous term. In the *Fibbonaci* sequence $a_1 = a_2 = 1, a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$ we need to know the two previous terms to determine the next one.

reduced row-echelon form Similar to row-echelon form with additional fourth condition:
(4) Any column that contains a leading 1 has zeros everywhere else.
The matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

are in reduced row-echelon form.

reduction formula Any formula that allows to reduce an expression to a simpler form. Many trigonometric formulas could be considered reduction formulas (*double angle, half angle*, etc). Also, many integral formulas have the name reduction formula. Example of an integral reduction formula is:

$$\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx.$$

reduction of order A method of finding solutions of differential equations (with constant or variable coefficients) when one of the solutions of that equation is known. The method allows to use that solution and reduce the equation to an equation of lower order, which is easier to solve. Example: Suppose we need to solve the equation $y'' + p(t)y' + q(t)y = 0$ and we already know that $y_1(t)$ is a solution. To find the second solution we seek it in the form $y = v(t)y_1(t) = v(t)e^{-t}$, where v is a yet to be determined function. Calculating the first and second derivatives of y and substituting back into the equation (also remembering that y_1 satisfies that equation) we arrive to the simple equation

$$y_1 v'' + (2y_1' + py_1)v' = 0,$$

which is indeed an equation of the first order if we denote $v' = u$. Now, this equation is always solvable by the method of multiplying factors.

reduction to systems of equations Any differential equation of order $n > 1$ could be reduced to a system of n first order equations with same amount of unknown functions. For example, the second order equation $y'' - 5y' + 2y = 0$ could be written as a system

$$x_1' = x_2, \quad x_2' = -2x_1 + 5x_2$$

if we make simple substitutions $x_1 = y,\ x_2 = y'$.

reference angle For a given angle $\theta, -\infty < \theta < \infty$ in standard position, the reference angle (usually denoted by θ') is the acute angle formed by the terminal side of the given angle and the horizontal axis. Hence, if the terminal side is in the first quadrant, then the reference angle is equal to the given angle (possibly subtracted or added some multiple of $360°$ to make it acute). Using degree measure, here are the formulas to calculate the reference angles in cases when the given angle is between $0°$ and $360°$.
1) If $0 < \theta < 90°$, then $\theta' = \theta$;
2) If $90° < \theta < 180°$, then $\theta' = 180° - \theta$;
3) If $180° < \theta < 270°$, then $\theta' = \theta - 180°$;
4) If $270° < \theta < 360°$, then $\theta' = 360° - \theta$.

reflection of a function A function (and its graph) could be reflected with respect to any line of the plane. For a given function $f(x)$ the function $g(x) = f(-x)$ is its reflection with respect to the y-axis, and the function $h(x) = -f(x)$ is its reflection with respect to the x-axis. The reflection of the graph of the function $f(x)$ with respect to the line $y = x$ represents the graph of the inverse function $f^{-1}(x)$, if the inverse exists.

reflection transformation Or reflection operator, is a transformation that maps each vector into its symmetric image about some line or plane. Example: The operator $T : R^2 \to R^2$ that transforms any vector to its symmetric with respect to the y-axis is given by the *standard matrix*

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

region A collection of points, a set, usually in two- or three-dimensional *Euclidean space*. Regions are the sets where functions of two or more variables are defined. There are different types of regions depending on certain properties.
(1) An open region (also set) is the one that with every of its points contains also a disk (circle) completely inside the region. Example of an open set is the *unit circle* $x^1 + y^2 < 1$ without the circumference.
(2) A closed region is the one that contains all of its boundary points. For a plane region, for example, the point (x, y) is a boundary point if any circle with center at that point contains points that belong to the region and points that do not belong to it. An example of the closed region is the "closed" unit circle $x^2 + y^2 \leq 1$.
(3)Connected region is the one where any two points of that region could be connected by a curve entirely belonging to the region.
(4)Simply connected region is the one where every simple, connected curve encloses only points that are in the region. Roughly speaking, simply connected regions do not have any "holes" or more than one "piece".

regression line See least squares regression line.

regular polygon A *polygon* that has equal sides (and also equal inner angles). Examples are the square, regular pentagon, hexagon, octagon, etc.

regular singular point For series solutions of *linear differential equations* of the type

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

If for some point $x_0,\ P(x_0) = 0$ but the limits

$$\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)}, \ \lim_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$$

are both finite, then the point x_0 is a regular singular point and for the equation the series solution is still possible. See the main entry series solution.

regular transition matrix A transition matrix is regular if some integer power of it has all positive entries.

related rates If two or more variable quantities are related to each other then their changes are also related. If there is a formula to connect the rate of change of one variable to the rate of change of the other(s), then these rates are called related. Example: The volume of a ball (sphere) is related to the radius by the formula $V = 4\pi r^3/3$. When the radius starts increasing at the rate dr/dt, the volume also starts increasing and the relation is $dV/dt = 4\pi r^2 dr/dt$.

relative error If in some calculation an error is made then the difference between the exact value and the approximate value is the absolute error. The ratio of the absolute error and the the exact value is the relative error.

relative growth rate The *growth rate* is the measure of the change of a variable quantity in a unit time, or its instantaneous change. If we divide that rate by the quantity itself, the result will be the relative growth rate. For example, if P denotes the size of some population, then dP/dt is its growth rate and $\frac{1}{P}dP/dt$ is the relative growth rate.

relative maximum and minimum The same as local maximum and minimum.

remainder (1) If an integer (*dividend*) is divided by another (*divisor*) and the result is not an integer, then whatever is left over is the remainder. For example, when we divide 127 by 5, the result is 25 and the remainder is 2. In cases when division is possible without remainder, we say that the remainder is zero.
(2) If a polynomial is divided by another polynomial, the result is some third polynomial and whatever is left over is the remainder of the division. In the division

$$\frac{2x^3 - 3x^2 + x - 4}{x^2 + 3} = (2x - 3) + \frac{7x - 13}{x^2 + 3}$$

the polynomial $7x - 13$ is the remainder. The *degree* of the remainder is always lower than the degree of the divisor.

remainder theorem If a polynomial $f(x)$ is divided by a binomial $x - k$, then the remainder is $r = f(k)$.
As a result, it follows, that if the polynomial is divisible by the binomial, then the remainder is zero.

remainder of a series Let Σa_n be a numeric or functional series. Then the sum of the first n terms $s_n = \Sigma_{k=1}^{n} a_k$ is its nth partial sum and the infinite sum of the terms starting from $n+1$ is its remainder $R_n = \Sigma_{n+1}^{\infty} a_k$. In case the series is convergent, $R_n = s - s_n$.

remainder estimates for series For most of the numeric or functional series it is either very difficult or even impossible to find the exact value that's why we try to find their approximate values. The best way to do it is to take some partial sums of that series. In that case the part we "cut" from the series (the infinitely many terms from the point we stop) is the remainder of that series. To estimate the accuracy of the approximation it is essential to have good estimates for these remainders. Below are two examples of estimates of remainders:
(1) Suppose $f(k) = a_k$, where f is a continuous, positive, and decreasing function for $x \geq n$ and suppose that $\sum a_n$ is convergent. If R_n is the remainder of that series, then

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx.$$

(2) Let $s = \sum(-1)^{n-1} b_n$ and s_n is its partial sum. Assume also that $0 \leq b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$. Then $|R_n| = |s - s_n| \leq b_n$.
For the estimates for remainders of *power series* see *Taylor series*.

removable discontinuity For functions that are not continuous (have discontinuity) at some point $x = c$, it is sometimes possible to redefine them at that point so they become continuous. This kind of discontinuity is called removable. Examples: The function $f(x) = (x^2 - 1)/(x - 1)$ is not defined at the point $x = 1$ (because the denominator is equal to zero there). Now, we can redefine this function as

$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if} \quad x \neq 1 \\ 2 & \text{if} \quad x = 1 \end{cases}$$

and this function will be continuous everywhere. In the case of the function $f(x) = 1/x$, on the other hand, it is impossible to redefine the function so it becomes continuous at the point $x = 0$, hence the discontinuity at that point is not removable.

representation of functions Functions could be represented in many different ways, the most common being the verbal, graphical, as a list or table, and algebraic or analytic. For specific examples see the entry *functions*.

representation of vector Any vector in a *vector space* V could be represented as a *linear combination* of vectors of any *basis* of the space V. See corresponding entries for more details.

residual The least squares regression line gives the predicted values of the variable in the given interval that are not necessarily equal to the *observed values* of the variable. The difference between the predicted value and the observed (actual) value of the variable is the residual.

resonance Many natural systems (mechanical, electromagnetic, acoustic, etc.) have their specific vibration frequencies. If an outside force, acting periodically, is applied to the system, and its own frequency is close or identical to the system's frequency, then the phenomenon of resonance occurs. In that case the amplitude of vibration of the system increases significantly even with very small outside force. Mathematically this could be expressed by equation of the form

$$my'' + \gamma y' + ku = F_0 \cos \omega t,$$

where the right side represents the outside periodic force.

response variable In Statistics, another name for dependent or y-variable. See also *explanatory variable*.

revenue function A function (in business and economics) that represents the revenue from production (or sales) of x units. If $p(x)$ is the price (or demand) function, then the revenue function is given by the formula $R(x) = xp(x)$.

reversing order of integration When evaluating *double integrals* (and, more generally, *multiple integrals*) it is important to be able to reverse the order of integration. This means that one can integrate first by one variable and then by the second one or do integration in reverse order (with appropriate change of limits of integration). Exact conditions when this is possible are given by Fubini's theorem. Example:

$$\int_0^1 \int_x^1 \sin(y^2)dydx = \int_0^1 \int_0^y \sin(y^2)dxdy$$

$$= \int_0^1 y\sin(y^2)dy = \frac{1}{2}(1 - cos1).$$

rhombus The special case of *parallelogram*, when all four sides are equal.

Riemann integral Let $f(x)$ be a continuous function on the closed finite interval $[a, b]$. Let us divide the interval into n equal parts of size $\Delta x = (b - a)/n$. Denote the endpoints of these parts by $x_0(= a), x_1, x_2, \cdots, x_n(= b)$ and chose arbitrary points $x_1^*, x_2^*, \cdots, x_n^*$ in each of these smaller intervals $[x_{i-1}, x_i]$, $1 \le i \le n$. Then the Riemann integral of f on the interval $[a, b]$ is the limit

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x.$$

Equivalently, the integral can be defined with the sample points x_i^* chosen to be the left endpoints, right endpoints or the midpoints of intervals.
The sum in the definition of the integral is called a *Riemann sum*. See also *definite integral*.

right angle An *angle* that measures 90° in *degree measure* or $\pi/2$ in *radian measure*.

right triangle Any triangle that has one right angle. The other two angles in such a triangle are necessarily *acute*.

right circular cylinder A *cylinder* that has a circle as its base and the sides are perpendicular to the base.

right-hand derivative If a function is not *differentiable* at some point $x = c$, then the limit in the definition of *derivative* does not exists, i.e.

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$$

does not exist. In some cases, however, one sided limit might exist. If the *right-hand limit*

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$$

exists, then it is called right-hand derivative of f at the point $x=c$. See also *left-hand derivative.*

right-hand limit Let $f(x)$ be a function defined on some interval $[a,b]$ (also could be an *open interval*). Right-hand limit of f at some point c is the limit of $f(x)$ as the point x approaches c from the right, or, which is the same, as $x>c$. The notation is $\lim_{x\to c^+} f(x)$. The precise definition of right-hand limit (the $\varepsilon-\delta$ definition) is the following: The function f has right-hand limit at the point c and that limit is L, if for any real number $\varepsilon>0$ there exists $\delta>0$ such that $|f(x)-L|<\varepsilon$ as soon as $|x-c|<\delta$, $x>c$. See also *limit* and *left-hand limit.*

Roll's theorem This theorem states that if a *differentiable function* has the same values at the endpoints of some closed interval, then somewhere inside that interval the tangent line to this function should be horizontal. Formally:
Let f be continuous on some interval $[a,b]$ and differentiable inside the interval (a,b). If, additionally, $f(a)=f(b)$, then there is a number $c, a<c<b$, such that $f'(c)=0$.

root (1) Has the same meaning as the *radical.* Let $n\geq 2$ be an integer and a and b– some real numbers such that $b=a^n$. Then we write $a=\sqrt[n]{b}$ and call a the n-th root of b. Not all real numbers have real n-th roots. For example, the number -3 has no real square root because the square of any real number is positive (or zero). If a real number has a real root, then the one that has the same sign as the number is called the *principal n-th root.* To guarantee the existence of the roots, the following definitions are given:
(a) If n is even and $a\geq 0$, then the n-th root of a is the positive number b such that $b^n=a$. For the case of negative number a see *roots of a complex number*;
(b) If n is even and a is any real number, then the n-th root of a is the real number b such that $b^n=a$.
The following properties are valid for all radicals that make sense under the definition above:

$$1)\ \sqrt[n]{a^m}=(\sqrt[n]{a})^m,\quad 2)\ \sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab},$$

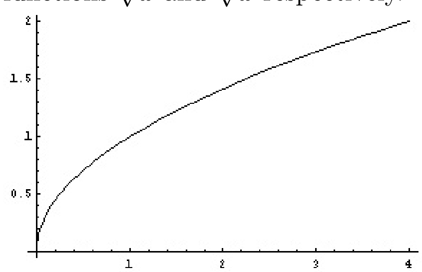
$$3)\ \frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\frac{a}{b}},\quad 4)\ \sqrt[m]{\sqrt[n]{a}}=\sqrt[nm]{a},$$

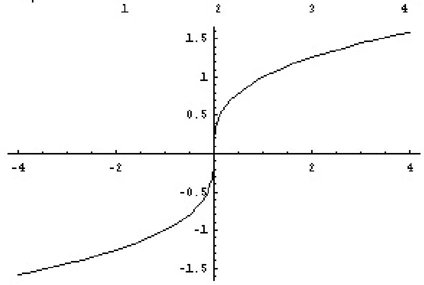
$$5)\ (\sqrt[n]{a})^n=a,$$

6) $\sqrt[n]{a^n}=|a|$ for n even; $\sqrt[n]{a^n}=a$ for n odd.

The roots are used also to define fractional exponents, because $a^{n/m}=\sqrt[m]{a^n}$ and they obey the same rules (properties) as the n-th roots.
(2) When solving algebraic equations, the term "root of the equation" is used as a synonym of the terms "solution of the equation" or "zero of the equation". See *zeros of the equation.*

root function The function $f(x)=\sqrt[n]{x}$, where n is a positive integer. If n is even, then the *domain* of this function is $[0,\infty)$ and for odd n the domain is the set of all real numbers. The graphs show the functions $\sqrt{x}$ and $\sqrt[3]{x}$ respectively.





roots of a complex number Any complex number z has exacly n roots of order n (denoted by $w_k,\ k=0,1,2,\cdots,n-1$). To find the nth root of a complex number $z=a+ib$, we write it in *trigonometric form* $z=r(\cos\theta+i\sin\theta)$ and use the formula

$$w_k=\sqrt[n]{r}\left[\cos\frac{\theta+2\pi k}{n}+i\sin\frac{\theta+2\pi k}{n}\right],$$

where $\sqrt[n]{r}$ means the positive nth root of the positive number r.
Example: To find the four fourth roots of the number $z = -2 + 2\sqrt{3}i$ we write $z = 4(\cos 120° + i \sin 120°)$ and $w_k,\ k = 0, 1, 2, 3$ are given by

$$\sqrt[4]{4}\left[\cos\frac{120° + 360°k}{4} + i\sin\frac{120° + 360°k}{4}\right],$$

from where we find $w_0 = \frac{\sqrt{6}}{2} + i\frac{\sqrt{2}}{2}, w_1 = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{6}}{2}, w_2 = -\frac{\sqrt{6}}{2} - i\frac{\sqrt{2}}{2}, w_3 = \frac{\sqrt{2}}{2} - i\frac{\sqrt{6}}{2}$.

roots of unity The nth roots of the number 1 considered as a complex number. We can write the number 1 in the trigonometric form as $1 = \cos 0 + i \sin 0$ and apply the formula for the *roots of a complex number* and get

$$w_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}, \quad k = 0, 1, \cdots, n-1.$$

The first root of unity is always $1 (w_0 = 1)$. Geometrically, the roots of unity are points located on the *unit circle* at the equal distance from each other.

root test A method of determining the absolute convergence or divergence of a (numeric or functional) series.
(a) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = a < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.
(b) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = a > 1$, then the series is divergent.
(c) In the case $a = 1$ the test cannot give a definite answer.

root law for limits If n is a positive integer, then

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)}.$$

In case when n is even, we need additionally assume that $\lim_{x\to c} f(x) > 0$. In the special case $f(x) = x$ we have $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{a}$.

rotation operator Also called rotation transformation. The operator in the *Euclidean space* that rotates each vector of that space with respect to some fixed point or some fixed axis by some specific angle. The rotation operator in the plane corresponding to rotations about the origin to the angle θ is given by the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

There are more rotation operators in three dimensional Euclidean space, corresponding to rotation about the origin or about some axis. for example, the counterclockwise rotation about the positive x-axis through an angle θ is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Similar matrices work for rotations about y-axis or z-axis.

round-off error Also called rounding error. The difference between the actual (exact) value and the numeric approximation value. Round-off errors are inevitable when representing *irrational* (and some rational) numbers because their numeric value contains infinitely many digits (too many digits, respectively). Example: The exact decimal value of the number $\sqrt{2}$ contains infinitely many digits (and, as a result, will never be known). Depending on situation we may use the numeric approximations 1.41, 1.4142, 1.414213562, etc. The round-off errors for these values are approximately 0.0042, 0.00001356, 0.000000000373.

row-echelon form A matrix is in the row-echelon form if the following conditions are satisfied:
(1) If a row does not consist of zeros only, then the first non-zero element is 1 (leading 1);
(2) Rows consisting of zeros only are grouped at the bottom;
(3) In two non-zero rows, leading 1 in the bottom row is to the right of the leading 1 of the row above.
The matrices

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

are in the row-echelon form.

row equivalence If a matrix A can be obtained from another matrix B by a finite sequence of elementary row operations, then matrix B can be obtained from matrix A by the same number of elementary row operation. Because every elementary row operation is *invertible*, the second sequence of matrices is just the inverses of the first sequence (in reverse order). These kind of matrices are called row equivalent.

row reduction The process of applying the Gauss or Gauss-Jordan elimination procedure to a given matrix. This process is useful, in particular, in evaluating the *determinants* of matrices.

row space For an $m \times n$ matrix A the vectors formed by the m rows of that matrix are called row vectors and the *subspace* of the *Euclidean space* R^n spanned by that vectors is the row space of the matrix. The row spaces of *row equivalent* matrices are the same.

ruled surface A surface that could be generated by the motion of a straight line. Examples of ruled surfaces are *cylinders, cones, hyperbolic paraboloids.*

S

saddle point Suppose the point (a, b) is a critical point for a function $f(x, y)$. If it is neither a maximum nor minimum value for the function, then the corresponding point on the graph of f (a point on the surface determined by the function) is called a saddle point.

sample A part of the *population* gathered with the intention of getting information about the population. The process of gathering samples is called sampling. In order to present statistical (and scientific) value all samples should be random. See the entry *simple random sample.* In addition to simple random sample we have the following special methods of sampling:
1) Probability samples give each element of the population equal chance to be chosen;
2) In stratified random sample we first divide population into groups of similar objects (stratas) and then used the simple random sample in each of these groups. The results should be put together;
3) In systematic samples we chose every nth element of the population.
There are some other methods of sampling all of which assure the un-biased character of the sample.

sample mean The mean (average) of all sample values. If $x_1, x_2, \cdots, x_n$ are the sample values then the mean is

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}.$$

sample space When considering a random phenomena, the collection of all possible outcomes is the sample space. For example, when rolling a dye, the sample space is the collection of the outcomes 1, 2, 3, 4 ,5, and 6.

sample standard deviation The square root of the *sample variance*: $s = \sqrt{s^2}$. See also *standard deviation.*

sample variance If $x_1, x_2, \cdots, x_n$ are all the values of the sample and $\overline{x}$ is the *sample mean*, then the variance is given by the formula

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}.$$

There is another formula, called shorthand formula, that is sometimes more convenient to use:

$$s^2 = \frac{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}{n(n-1)}.$$

See also *variance*.

sampling distribution Suppose we have a distribution from large population. Chose a simple random sample of size n from that population. After that we find the *mean* of that sample and denote it by $\bar{x}_1$. Then we do the same process the second time and denote the mean of the second sample of size n by $\bar{x}_2$. If we repeat this process many times we will get a sequence, or set of the values $\{\bar{x}_1, \bar{x}_2, \bar{x}_3, \cdots\}$. The distribution formed from these values is the sampling distribution. According to the *Central limit theorem*, the sampling distribution is always normal (more precisely, approaches to normal, if n increases). The same theorem also establishes relationship between the means and standard deviations of these distributions.

sawtooth wave The function defined as $f(t) = t,\ 0 \leq t < 1$ and then repeated periodically with the period 1: $f(t+1) = f(t)$.

scalar The same as a *constant*. A quantity that does not change. Real and complex numbers are scalars.

scalar equation of a plane The same as *equation of the plane* and in three dimensional space is given by

$$ax + by + cz + d = 0.$$

scalar field Or the field of scalars. In choosing the scalars (constants) to use in different situations, we usually choose among the fields of *rational numbers, real numbers* or *complex numbers*. When we need a real *vector space*, then the choice is the field of real numbers. Also we consider polynomials using coefficients from the field of rational numbers.

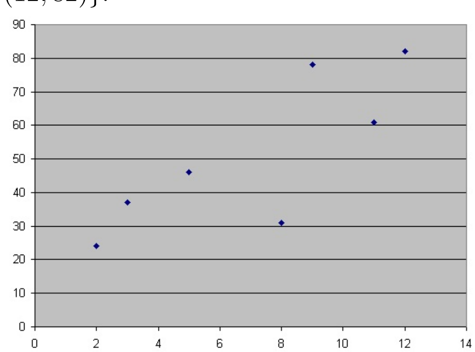
scalar product Has the same meaning as the *inner product*. Let $\mathbf{x}, \mathbf{y}$ be two vectors. Then
(1) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$; (2) For any real a, $\langle a\mathbf{x}, y \rangle = a\langle \mathbf{x}, y \rangle$; (3) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $\mathbf{x} = 0$.

scalar multiplication The operation of multiplying vectors or matrices by a scalar. In the case of matrices, to multiply by a scalar means to multiply each entry of the matrix by the same constant. Hence,

$$2 \cdot \begin{pmatrix} 1 & 0 & -2 \\ -1 & 3 & 4 \\ 5 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -4 \\ -2 & 6 & 8 \\ 10 & -4 & 0 \end{pmatrix}.$$

In the case when vectors are given in the column or row form, then scalar multiplication works exactly as in the case of matrices. If the vectors are given in an abstract vector space, then scalar multiplication is defined in the form of axioms (part of the axioms of a *vector space*):
(1) If k is a scalar and $\mathbf{u}$ is in the space V, then $k\mathbf{u}$ also is in V;
(2) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$;
(3) $(k + l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$;
(4) $k(l\mathbf{u}) = (kl)\mathbf{u}$.

scatterplot Also called scatter diagram or scatter graph. If the statistical data comes in pairs (*paired data*), then they could be viewed as *ordered pairs* and presented on the *Cartesean plane* as a number of points. The result is called scatterplot. The figure shows the scatterplot of paired data $\{(2,24), (3,37), (5,45), (8,31), (9,78), (11,61), (12,82)\}$.



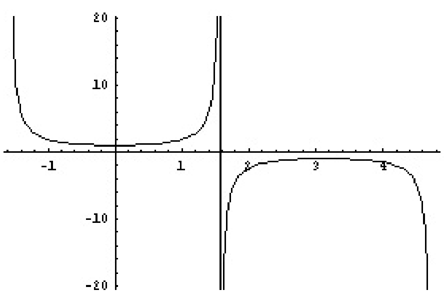
scientific notation Notation used to write very large or very small numbers in a more compact form.

In this form the number is written as a number between 1 and 10 multiplied by some power of 10. Examples: (1) The number 4250000000 can be written as 4.25×10^9. (2) The number 0.0000716 could be written as 7.16×10^{-5}.

secant function One of the six trigonometric functions. Geometrically, the secant of an angle in a right triangle, is the ratio of the *hypothenuse* of the triangle to the *adjacent side*. Also could be defined as the reciprocal of the cosine function. The function $\sec x$ could be extended to all real values exactly as the $\cos x$ function. The domain of $\sec x$ is all real values, except $x = \pi/2 + \pi n$, n any integer, and the range is $(-\infty, -1] \cup [1, \infty)$. $\sec x$ is 2π-periodic.
One of the *Pythagorean identities* relates the secant function to the tangent function: $1 + \tan^2\theta = \sec^2\theta$. The following are the calculus related formulas:

$$\frac{d}{dx}(\sec x) = \sec x \tan x,$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C.$$



secant line A line that connects two points on some curve. The segment of the secant line between these two points is the *chord*.

second derivative test A test that allows to find local maximums or minimums and substitutes the *first derivative test* in cases when the second derivative at the *critical points* exists.

Assume $f''(x)$ is continuous near the point c.
(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at the point c;
(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at the point c.

sector of a circle A part of a circle that is bounded by two radiuses and the arc of the circumference connecting the endpoints of radiuses.

semi-circle One half of a circle. Also could be viewed as a sector formed by two radiuses making $180°$ angle.

separable differential equations In the most general setting, the equation is separable if it could be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0.$$

Another way of describing separable equations is to write them in the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}.$$

To solve this equation we write $g(y)dy = f(x)dx$ and integrate to find the general solution

$$\int g(y)dy = \int f(x)dx.$$

Example: Solve the equation $dy/dx = x^3/y^2$. Solution: $y^2 dy = x^3 dx$,

$$\int y^2 dy = \int x^3 dx,$$

$y^3/3 = x^4/4 + C$, and $y = \sqrt[3]{3x^4/4 + 3C} = \sqrt[3]{3x^4/4 + K}$.

sequence A sequence is a list of numbers, functions, vectors, etc. written in some order. Terms (or members) of a sequence are numbered using the set of natural or whole numbers or all integers.
(1) Numeric sequences. A sequence could be viewed as a function of natural (whole, integer) variable. Hence, we write $f(1) = a_1, f(2) = a_2, \cdots$. A sequence is finite if it has only finite number of terms. Otherwise it is called infinite. A sequence $\{a_n\}$ is bounded, if there is a number $M > 0$ such that $|a_n| \leq M$ for all n. $\{a_n\}$ is called increasing if any next term is greater than the previous one: $a_{n+1} > a_n$. Similarly, it is called decreasing if $a_{n+1} < a_n$. The general name for increasing and decreasing sequences is monotonic.

Limits for sequences are defined like limits for functions. A sequence $\{a_n\}$ is said to have a limit L as $n \to \infty$ if for any $\epsilon > 0$ there is a $N > 0$ such that $|a_n - L| < \epsilon$ as soon as $n > N$. We write $\lim_{n\to\infty} a_n = L$. A sequence that has a limit as $n \to \infty$ is called convergent. Otherwise it is divergent. In case when the sequence is divergent in a specific way that it increases without bound, we say that it goes to infinity. Formally, $\lim_{n\to\infty} a_n = \infty$ means that for any $M > 0$ there is an integer N such that $a_n > M$ as $n > N$. The case $\lim_{n\to\infty} a_n = -\infty$ is defined similarly.
(2) Sequences of functions could be treated similar to numeric sequence. Again, we have the notions of bounded , increasing or decreasing sequences. The most common functional sequences are sequences of monomials $\{x^n\}$ and trigonometric functions $\{\sin nx,\ \cos nx\}$.

series The sum of the terms of a finite or infinite, numeric or functional *sequence*. Accordingly, we have numeric or functional series.To write the series we use the *summation (sigma) notation*.
1) Numeric series. The series $\sum_{n=0}^{\infty} a_n$ is called convergent, if the sequence of its partial sums $S_n = \sum_{k=0}^{n} a_k$ is a convergent sequence. The precise definition is: We say that the series $\sum_{n=0}^{\infty} a_n$ is convergent and converges to some finite number S, if for any given $\epsilon > 0$ there exists a natural number N such that $|S_n - S| < \epsilon$ as soon as $n \geq N$. In order to find out if a series is convergent or not, various convergence tests may be used. For positive series see *integral test, comparison test, lomit comparison test.* For *alternating series* see the *alternating series test.* Finally, for *absolute convergence* of the series see the *ratio test* or the *root test.*
2) Functional series. If the terms of the series consists of functions, then it is called functional series. A functional series may be convergent for some values of the variable and divergent for other values. The definition of convergence of a functional series at any point is exactly the same as the definition of the convergence of a numeric series. The most common functional series are *power series* and *trigonometric series.* Besides them, other functional series are also considered (such as Bessel series or series of Laguerre polynomials). See also *binomial series, Taylor series, MacLaurin series.*

series solution For linear differential equations. Assume that we are solving the equation

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = 0,$$

where the coefficients $a_1(x), a_2(x), \cdots, a_m(x)$ are *analytic functions*, hence can be represented by their *Taylor-MacLaurin* series. The idea behand the series solutions of this equation is that it is natural to look for analytic solutions, i.e. solutions, represented by power series. This idea extends also to some cases when the coefficients are not analytic but have some "regularity" properties. The method works the best when the coefficients $a_i(x)$ are polynomials, or even better, if they are monomials. We will demonstrate this method for the case of second order equation

$$P(x)y'' + Q(x)y' + R(x)y = 0. \qquad (1)$$

A point $x = x_0$ is called *ordinary point* for this equation, if $P(x_0) \neq 0$, otherwise it is called a singular point.
(A) The case of the ordinary point. Dividing the equation by $P(x)$, we will have a simpler form near the ordinary point x_0 (because $P(x_0) \neq 0$)

$$y'' + p(x)y' + q(x)y = 0 \qquad (2)$$

and will look for a solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where the coefficients a_n are still to be determined. If we calculate the first and second derivatives of this unknown function and also write power series representations of functions $p(x)$ and $q(x)$ around the point x_0, then we will get

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}$$

$$+ \left(\sum_{j=0}^{\infty} p_j (x-x_0)^j \right) \left(\sum_{n=1}^{\infty} n a_n (x-x_0)^{n-1} \right)$$

$$+\left(\sum_{i=0}^{\infty} q_i(x-x_0)^i\right)\left(\sum_{n=0}^{\infty} a_n(x-x_0)^n\right)=0.$$

Finally, if we perform all multiplications and additions, equate resulting coefficients of all powers to zero (because on the right side we have the identically zero function), then we will find expressions for coefficients a_n we the help of the known coefficients p_j, q_i of the functions $p(x)$ and $q(x)$.
Example: Find the series solution of Airy's equation

$$y'' - xy = 0, \qquad -\infty < x < \infty.$$

Proceeding as described before and using the fact that the functions $p(x) = 0$ and $q(x) = x$ are very simple, we will have series solution presentation centered at $x = 0$ (here the summation index is changed to a more convenient form):

$$2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n = \sum_{n=1}^{\infty} a_{n-1}x^n.$$

Now, clearly $a_2 = 0$, because the right side does not have a constant term and from the recurrence relation

$$(n+2)(n+1)a_{n+2} = a_{n-1}$$

that follows from equating the coefficients of equal powers, we have also $a_5 = a_8 = \cdots = 0$, or $a_{3n-1} = 0$, $n = 1, 2, 3, \cdots$. Similarly, we find that

$$a_{3n} = \frac{a_0}{2\cdot 3\cdot 5\cdot 6\cdots(3n-1)(3n)}$$

and

$$a_{3n+1} = \frac{a_1}{3\cdot 4\cdot 6\cdot 7\cdots(3n)(3n+1)}$$

and the solution of the equation (2)is given by the formula

$$y = a_0\left[1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} + \cdots\right]$$

$$+a_1\left[x + \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} + \cdots\right].$$

This solution depends on constants a_0 and a_1. If we need to find a unique solution then it is necessary to have two *initial or boundary values* given.

(B) The case of the singular point. If the function $P(x)$ from the equation (1) above becomes zero at any point, then the described method does not work. There is however an exceptional case where a similar method gives a satisfactory solution. If $P(x_0) = 0$ and

$$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)},\ \lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)}$$

are both finite, then the point x_0 is called *regular singular point*. Now assuming for simplicity that the point $x = 0$ is a regular singular point for the equation (1) we divide that equation by $P(x)$ and then multiplying everything by x^2 get the simplified equation

$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0. \quad (3)$$

Here we are seeking solutions in the form

$$y = x^r\sum_{n=0}^{\infty} a_nx^n,$$

where both the coefficients a_n and the index r are to be determined. Proceeding as in the case of ordinary points and equating the coefficients of equal powers we will have a recurrence relationship for the coefficients and the index:

$$F(r+n)a_n + \sum_{k=0}^{n-1} a_k[(r+k)p_{n-k} + q_{n-k}] = 0,$$

where $F(r) = r(r-1) + p_0 + q_0$.

Theorem. (a) Let $r_1 \geq r_2$ be the two real roots of the *indicial equation* $F(r) = 0$. Then there exists a solution of the form

$$y_1 = |x|^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right],$$

where $a_n(r_1)$ are found from recurrence relation above.
(b) If $r_1 - r_2$ is not zero or a positive integer, then there exists a second linearly independent solution of the form

$$y_2 = |x|^{r_2}\left[1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right].$$

(c) If $r_1 = r_2$, then the second solution is given by

$$y_2 = y_1(x)\ln|x| + |x|^{r_1}\sum_{n=1}^{\infty} b_n(r_1)x^n.$$

(d) If $r_1 - r_2 = N$ is a positive integer, then

$$y_2 = ay_1(x)\ln|x| + |x|^{r_2}\left[1 + \sum_{n=1}^{\infty} c_n(r_2)x^n\right].$$

The coefficients $a_n(r_1)$, $b_n(r_1)$, $c_n(r_2)$ and the constant a can be determined by substituting the solutions into the equation (3). The constant may equal zero and the corresponding term will be missing.

sets One of the most basic notions of mathematics. By a set we understand any collection of objects of any nature: numbers, people, countries, stars, etc. Objects of a set are called its elements or members. Two sets A and B are considered equal (more precisely, identical) if they have exactly the same elements. A set B belongs to another set A (is a subset of A, notation: $B \subset A$), if all elements of B are also elements of A, but the opposite is not necessarily true. For a set A, its complement $\overline{A}$ is the set of all elements that do not belong to A. For example, in the larger set of all *real numbers*, the complement of all rational numbers is the set of all irrational numbers. For the operations of intersection and union of sets see corresponding entries.

shift of index of summation In *summation notation* for functions it is sometimes important or convenient to "shift" the index of summation to be able to combine two or more sums. For example, the sum

$$\sum_{k=1}^{n} a_{k-1}x^k$$

could be written also as $\sum_{k=0}^{n-1} a_k x^{k+1}$ using the substitution $m = k - 1$ and then returning to the index k because the notation does not make the sum any different.

shifted conics The three major conic sections (*ellipses, parabolas, hyperbolas*) may be shifted horizontally or vertically or both. The result is a shifted conic. See the equations in the corresponding entries.

shift of a function The process of moving the graph of a function horizontally or vertically or both. Algebraically the vertical shift is achieved by adding or subtracting a number from the function: $g(x) = f(x) + c$ is the vertical shift if the function$f$. If $c > 0$ then the function is shifted up and if $c < 0$ then the shift is down. The horizontal shift of the function f is given by the formula $g(x) = f(x - c)$. If $c > 0$ then the function shifts to the right and if $c < 0$ it shifts to the left. For example, the graph of the function $g(x) = (x + 3)^2 + 2$ could be found from the graph of the parabola $f(x) = x^2$ by shifting it left 3 units and shifting up 2 units.

sieve of Erathosthenes The ancient method of finding prime numbers. Suppose we need to find all the prime numbers up to some number N. Make the list (table) of all these numbers and start eliminating the numbers that are not prime as follows: First throw away the number 1 because it is not prime. Next, leave 2 because it is prime but throw away all multiples of 2 (all even numbers) up to N. In the third step leave 3 but eliminate all the multiples of 3 that are still in the list (all even numbers that are multiples of 3 were eliminated in the previous step). Continuing this way we will throw away all non-prime numbers and what is left would be all prime numbers up to N. During this process our number list gets lots of "holes" which is the reason for the name "sieve".

sigma notation The same as summation notation.

signed elementary product An *elementary product* multiplied by +1 or -1. Used to calculate determinants.

signed number A term that is sometimes used to indicate negative number, as numbers preceded by the "minus" sign.

significance level Also called alpha level and denoted by the Greek letter α. For tests of significance, a numeric level such that any event with probability below that is considered rare. It is set arbitrarily but the usual choices are 0.1, 0.05 and 0.001. The choice of the significance level depends on the importance of the problem.

significance tests See test of significance.

similar terms The same as like terms.

similar matrices Square matrices A and B are similar if there exists an *invertible matrix* P such that $B = P^{-1}AP$. similar matrices have many properties and characteristics in common. In particular, the *determinants, traces, eigenvalues, nullity* of similar matrices are the same. If the two matrices are the *standard matrices* of some transformations T_1 and T_2 then the transformation bringing one to the other is called similarity transformation

simple curve A curve is simple if it does not intersect itself, except maybe the endpoints. If the curve is given by the *parametric equations* $x = f(t), y = g(t), a \leq t \leq b$, then being simple means $(f(c), g(c)) \neq (f(d), g(d))$ for $c, d \neq a$ and $c, d \neq b$. If the endpoints coincide, i.e., if $f(a) = f(b)$ and $g(a) = g(b)$, then we call it a simple closed curve.

simple region A term sometimes used to describe regions that lie between the graphs of two functions. Similarly, simple solid region is a region that lies between two surfaces described by functions of two variables.

simple eigenvalues See eigenvalue of matrix.

simple harmonic motion If a particle or an object moves (oscillates) according to the equation $y = A\cos(\omega t - \delta)$ then this kind of motion is simple harmonic. Here A is the amplitude, $\omega/2\pi$ is the frequency and δ is the phase shift.

simple interest If the bank pays interest on the principal amount only then the interest is called simple. Mathematically, if P is the amount of the money invested at the rate r yearly, then after n years that principle will become $P + nrP = P(1 + nr)$. See also *compound interest.*

simple random sample One of the most important notions in statistics. A simple random sample of size 2 from a given *population* P is a *sample* where every possible pair of values has equal chance to be chosen. A simple random sample of size 3 is a sample from the population where every possible triple of values from P has equal chance to be chosen. More generally, a simple random sample of size n is a sample where every possible set of n values from P has exactly the same chance to be chosen.

simply connected region If any *simple closed curve* in the region contains only points from the region, then it is called simply connected. Another, simpler definition says, that if any two points in the region can be connected by a continuous curve that is completely inside the region, then it is simply connected. In other words, the region basically consists of one "piece".

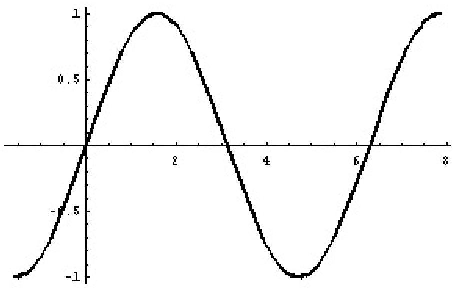
Simpson's rule One of the methods of *approximate integration.* To calculate the approximate value of the integral $\int_a^b f(x)dx$ using Simpson's rule we divide the interval $[a, b]$ into even number n equal intervals of the length $\Delta x = (b - a)/n$ with the points $x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \cdots, x_n = b$. Then we have the approximation

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2)$$

$$+4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)].$$

For the degree of accuracy of this approximation see error estimate for Simpson's rule.

sine function One of the six trigonometric functions. Geometrically, the sine of an angle in a right triangle is the ratio of the *opposite side* to the *hypotenuse* of the triangle. More general approach allows to define $\sin x$ function for any real x: Let $P = (a, b)$ be any point on the plane other than the origin and θ is the angle formed by the positive half of the x-axis and the terminal side, connecting the origin and P. Then $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$. Next, after establishing one-to-one correspondence between angles and real numbers, we can have the sine function defined for all real numbers. The range of $\sin x$ is $[-1, 1]$ and it is 2π-periodic.



The sine function is related to other trigonometric functions by various identities. The most important is the *Pythagorean identity* $\sin^2 x + \cos^2 x = 1$. The derivative and indefinite integral of this function are:

$$\frac{d}{dx}(\sin x) = \cos x, \quad \int \sin x dx = -\cos x + C.$$

sine integral function The function defined by the integral

$$Si(x) = \int_0^x \frac{\sin t}{t} dt.$$

The integrand cannot be expressed with elementary function and the values of this function are found by approximate (numeric) integration.

singular matrix The opposite of an *invertible matrix.* For a square matrix A to be singular means there is no matrix B such that $A \cdot B = I$, where I is the *identity matrix.* An easy criteria for the singularity is that the determinant of a singular matrix is zero.

singular point A point where the function cannot be defined because it has infinite limit (from the left, from the right, or both). The point $x = 1$ is singular for the function $f(x) = 1/(x-1)^2$ because

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty.$$

A function has singular point at infinity if $\lim_{x\to\infty} f(x) = \pm\infty$. For the use of the term singular in differential equations see *irregular singular point, regular singular point.*

size of a matrix Size of a matrix is the number of rows and columns it has. So, if the matrix has 4 rows and 5 columns then we say that it is of size 4×5.

skew lines Two lines in three dimensional space are called skew lines if they are not parallel and do not intersect. Skew lines do not lie in any plane.

skew-symmetric matrix A square matrix is skew-symmetric if it is equal to the opposite of its *transpose*: $A = -A^T$.

slant asymptote An *asymptote* that is neither horizontal nor vertical. For exact definition see the entry asymptote.

slope One of the most important characteristics of a line on a plane. Shows the level of "steepness" or "flatness" of the line. If the points (x_1, y_1) and (x_2, y_2) are on the line, then, by definition, the slope of the line is the number $m = (y_2 - y_1)/(x_2 - x_1)$. For a horizontal line the slope is zero and for a vertical line the slope is not defined because the denominator in the definition becomes zero.

slope field Same as direction field.

slope-intercept equation of a line If the *slope* m and the y-intercept b of the line are known, then the equation of the line is given by the formula

$$y = mx + b.$$

Example: If the slope of the line is $m = 2$ and the y-intercept is the point $(0, -1)$, then the equation of the line is $y = 2x - 1$. See also *line.*

smooth curve A curve created by a *smooth function* or smooth functions.

smooth function Usually, a function that has continuous derivative. In some other settings smooth function might mean a function that has more than one derivative. In particular, infinitely differentiable functions are also called smooth. The notion of the smooth function extends also to functions of several variables.

smooth surface A surface given by a differentiable function $f(x, y)$ of two variables. Geometrically this means that the surface has no "corners" or "edges".

solid Usually, a three dimensional object, geometric figure. For example, solid angle means the figure that we receive when we connect a fixed point with all the points of a closed simple curve in space. Solid of revolution means a solid that is received by rotating a plane region about some line, such as coordinate axis.

solution Depending on the type of equation the solution would be a number, set of numbers, set of ordered pairs or ordered triples, a matrix, a vector, a function or set of functions, etc. All these solutions have the same common property: substitution into the equation (or system of equations) results in a true statement, an identity. For details of how to find solutions of equations see the articles *quadratic equation, cubic equation, quartic equation, rational equations, radical equations, systems of linear equations, linear ordinary differential equation* and many others where the solutions of the corresponding equations are presented.

solution curve The graph of the solution of a differential equation.

solution set The set of all solutions of the given equation.

solution space Let $A\mathbf{x} = \mathbf{b}$ be a system of linear equations. Each solution $\mathbf{x}$ of this equation is called solution vector. The set of solutions forms a linear space called solution space.

solving triangles To solve a triangle means to find one or more elements (side or angle) of the triangle knowing the other elements. To solve a triangle we need to know at least three elements (including at least one side), because otherwise the problem becomes impossible to solve. The follwing four cases are possible in solving triangles: (1) One side and two angles are known (AAS or ASA); (2) Two sides and one angle not formed by them is known (SSA); (3) Two sides and the angle formed by them are known (SAS), and (4) Three sides are known (SSS). The first two cases are handled by the Law of sines and the other two cases are solved by the Law of cosines. Also, the second case is called ambiguous, because it not always results in definite solution. For that case see the separate entry ambiguous case. In the following examples the angles are denoted by A, B, C and the sides are denoted by a, b, c and we follow the convention that side a is opposite to angle A, side b is opposite to angle B and side c is opposite to angle C.

(1) The case AAS. Let $C = 102.3°,\ B = 28.7°,\ b = 27.4$. First we find the angle A using the fact that the sum of all angles is $180°$: $A = 49.0°$. Then using the Law of sines we have $a = b\sin A/\sin B \approx 43.06$. Similarly, $c = b\sin C/\sin B \approx 55.75$ and the triangle is solved.

(2) The case SAS. Let $A = 115°,\ b = 15,\ c = 10$. First, we use Law of cosines to find the side a : $a^2 = b^2 + c^2 - 2bc\cos A \approx 451.79$ and $a \approx 21.26$. After this, we have the choice of using either the Law of cosine again or the Law os sines to find one of the missing angles. Law of sines will give $\sin B = (b/a)\sin A \approx 0.63945$ and $B \approx 39.75°$. Now, $C \approx 25.25°$ and the triangle is solved.

(3) The case SSS. Let $a = 8,\ b = 19,\ c = 14$. Using the Law of cosines we have, for example,

$$\cos B = \frac{a^2 + b^2 - c^2}{2ac} \approx -0.45089$$

from where $B \approx 116.80°$. Now, the rest follows as in the previous case and we can find the remaining algles as before: $A \approx 22.08°,\ C \approx 41.12°$. The solution is complete.

space In mathematics the term has different meanings depending on context. Most commonly space means the three dimensional space we live in which is mathematically the three dimensional *Euclidean space*. In other cases space refers to *vector space*. See corresponding entries for more details.

space curve A curve in three dimensional space could be given by different methods. Suppose t is a variable on some interval $[a, b]$ and the functions $f,\ g,\ h$ are defined and continuous on that interval. Then the set of all points (x, y, z) in space where

$$x = f(t), \quad y = g(t), \quad z = h(t)$$

is a space curve given parametrically by these equations.

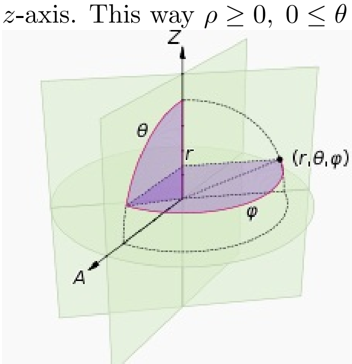
The same curve could be also given as a vector with the equation

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the *standard bases* in the three dimensional space.
Space curve is called continuous, differentiable, etc. if the functions f, g, h have the same properties.

sphere One of the most common geometric figures. The term has two meanings, to indicate the geometric solid or its boundary. In the first case the term ball is more commonly used. The equation of the ball (solid sphere) of radius r centered at the points (a, b, c) is $(x-a)^2+(y-b)^2+(z-c)^2 \le r^2$. The equation of its boundary (the sphere) is $(x-a)^2+(y-b)^2+(z-c)^2 = r^2$.

spherical coordinates One of the most common coordinate systems in three dimensional space along with the *Cartesian* and *cylindrical* systems. It is the three dimensional analog of the *polar* coordinate system on the plane. The points in spherical coordinate system are determined by three parameters: the distance from the origin (usually denoted by ρ) and two angles. The first angle, denoted by θ is measured on the xy-plane from the positive x-axis and the second angle, denoted by ϕ, is measured from the positive z-axis. This way $\rho \ge 0$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.



The relations between the spherical and rectangular coordinates are given by the formulas

$$x = \rho\cos\theta\sin\phi,\ y = \rho\sin\theta\sin\phi,\ z = \rho\cos\phi.$$

spiral of Archimedes See Archimedian spiral.

spread of a distribution The meaning of this term depends on situation but as a rule the spread of any distribution is described by its *standard deviation* or *variance.*

square matrix A matrix such that the number of rows is equal to the number of columns. There are many matrix operations that are possible for square matrices but not for other type of matrices. For example, the notions of *determinant* and *inverse of a matrix* do not make sense if the matrix is not square. For other entries specifically applied to square matrices see *nilponent matrix, normal matrix, skew-symmetric, symmetric, unitary matrices.*

square root The *radical* of the second order. The square root of a non-negative number $a \ge 0$ is defined to be the non-negative number $b \ge 0$ such that $b^2 = a$. The notation for the square root of the number a is $\sqrt{a}$. If the number a is negative then its square root is not a real number, it is *imaginary.* See also the entry *root* for more details.

squeeze theorem Assume that three functions $f(x), g(x), h(x)$ satisfy the inequalities $f(x) \le g(x) \le h(x)$ near some point $x = c$ but not necessarily at that point. Assume also that

$$\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$$

then

$$\lim_{x\to c} g(x) = L.$$

Example: Evaluate the limit

$$\lim_{x\to 0} x\cos(1/x).$$

Because the function $\cos(1/x)$ has no limit at zero, the usual *product law* for limits does not apply. However, since $|\cos(1/x)| \le 1$,

$$-x \le x\cos(1/x) \le x.$$

Now, $\lim_{x\to 0} x = \lim_{x\to 0}(-x) = 0$ and by the squeeze theorem our limit is also zero.

standard basis vectors In many common *vector spaces* sets of vectors that form a basis are considered to be standard. For example, in the *Euclidean space* R^n the set of vectors $\mathbf{e_1} = (1, 0, 0\cdots, 0)$, $\mathbf{e_2} = (0, 1, 0, \cdots, 0), \cdots, \mathbf{e_n} = (0, 0, 0, \cdots, 1)$ is considered (and called) the standard basis. In the space P_n of all polynomials of degree n or less the set of functions $(1, x, x^2, \cdots, x^n)$ is the standard basis.

standard deviation For a set of values $x_1, x_2, x_3, \cdots, x_n$ with the mean μ, the standard deviation is defined by the formula

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n-1}}.$$

The square of this expression is called *variance*. There are other, sometimes easier formulas for calculating standard deviation. See *sample standard deviation*.

standard equations A term used in different situations with specific meanings. For example, the standard equation of a line is $ax+by=c$, or the standard equation of an ellipse centered at the point (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

standard error In Statistics the *standard deviation* of the *sampling distribution* is usually unknown. To estimate it we use values of the sample and call it sampling error because it almost never equals the actual standard deviation. The standard error for the *sample mean* $\bar{x}$ is given by the formula $SE(\bar{x}) = s/\sqrt{n}$, where s is the *sample standard deviation* and n is the size of the sample. The standard error for proportions is

$$SE(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}},$$

where $\hat{p}$ is the *sample proportion* and n is the sample size.

standard matrix Suppose T is a linear transformation from the *Euclidean space* R^n to R^m. Then the vectors $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ are transferred to vectors $\mathbf{y} = (y_1, y_2, \cdots, y_m)$ by linear equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = y_m$$

The matrix $A = [a_{ij}]$, $1 \leq i \leq m$, $1 \leq j \leq n$ formed by the coefficients of these equations is called the standard matrix of the transformation T. The notion of the standard matrix could be extended also to transformations in general linear vector spaces.

standard normal distribution The special case of the normal distribution, when the *mean* is equal to 0 and the *standard deviation* is equal to 1. If the *random variable* describing some normal distribution with the mean μ and standard deviation σ is x then the variable $z = (x-\mu)/\sigma$ describes the corresponding standard normal distribution. The graph of the standard normal distribution is also the *normal density curve* centered at 0.

standard position Usually refers to angles or vectors. In the case of angles, the standard position is when the *vertex* is located at the origin and the *initial side* coincides with the positive side of the x-axis. In the case of the vectors the standard position is when the *initial point* coincides with the origin. See also *angles, vectors*.

standardized value The same as the z-score.

statistical inference Also called inferential statistics. The use of statistical information (data) to draw conclusions about larger population.

statistics One of the main branches of Mathematics. It is concerned with collecting and interpreting data. By other definitions, it is a mathematical science pertaining to the collection, analysis, interpretation or explanation, and presentation of data. Some other definitions of this science also include the fact that Statistics most of the time deals with random phenomena and needs to draw conclusions from randomly received data. See different entries about Statistics throughout this Dictionary.

stemplot Also called stem-and-leaf plot. An easy and fast way of establishing the form of the distribution for relatively small data set. For example, to organize the set of values $\{44\ 46\ 47\ 49\ 63\ 64\ 66\ 68\ 68\ 72\ 72\ 75\ 76\ 81\ 84\ 88\ 106\}$, we separate the "tens" digits and combine all the "ones" digits that correspond to each particular tens digit. The result is

```
 4 | 4 6 7 9
 5 |
 6 | 3 4 6 8 8
 7 | 2 2 5 6
 8 | 1 4 8
 9 |
10 | 6
```

step function The function $f(x) = [[x]]$ defined as the greatest integer less than or equal to x. Different variations of this function also may be called step function or staircase function.

Stokes' theorem One of the most important theorems of the multivariable Calculus. It could be viewed as the generalization of the *Fundamental Theorem of Calculus* and also of the *Green's theorem.* This theorem relates surface integrals of some functions with integrals on the boundary curve of that surface.
Stokes' theorem. Let S be an oriented piecewise-smooth surface bounded by a simple, closed, piecewise-smooth boundary curve C with *positive orientation.* Let $\mathbf{F}$ a vector field with continuously differentiable components in a region $G \subset R^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_S curl\mathbf{F} \cdot d\mathbf{S}.$$

For definitions of vector field and curl of a vector field see corresponding entries. See also *divergence theorem* for related results.

stretching a function One of the possible transformations of functions. The vertical stretch of the function f is given by the formula $g(x) = cf(x)$, where $|c| > 1$. In case when $|c| < 1$ the term squeezing or compression is used. The horizontal stretching is given by the formula $g(x) = f(cx)$ where $|c| < 1$. In case $|c| > 1$ the transformation is squeezing or compression.

straight angle An angle that is 180° in *degree measure* or π in *radian measure.*

straight line Or simply a line. One of the most important geometric figures (notions). A line is not defined but assumed to be understood as other fundamental geometric or mathematical notions. In algebra, a line is given by its equation. See the entry *line* for all the details.

submatrix Any part of a given matrix could be considered as a matrix, called submatrix.

subset A set B is called the subset of the set A (written $B \subset A$), if any element of B is also an element of A. Example: The set $\{2, 4, 5\}$ is the subset of the set $\{1, 2, 3, 4, 5\}$.

subspace A *subset* W of a vector space V is called a subspace if W itself is a vector space under the same rules of addition and scalar multiplication as for the space V.

substitution method or substitution rule. One of the main methods of evaluating indefinite or definite integrals. The essence of the method is to find some kind of substitution that will make the function under the integral sign simpler and easier to evaluate the integral. This method works only in cases when the function could be presented or viewed as a product of a composite function and the derivative of the function that is the argument of that function. Formally:
Let $u = g(x)$ be differentiable, then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Examples: (1) $\int \sqrt{3x-1}dx$. The substitution $u = 3x-1$ results in $du = 3dx$ and the integral transforms to

$$\int \sqrt{u}\frac{du}{3} = \frac{1}{3}\int u^{1/2}du$$

$$= \frac{2}{9}u^{3/2} + C = \frac{2}{9}(3x-1)^{3/2} + C.$$

(2) $\int \sin^4 x \cos x dx$. Here the substitution $u = \sin x,\ du = \cos x dx$ gives

$$\int \sin^4 x \cos x dx = \int u^4 du$$

$$= u^5/5 + C = \sin^5 x/5 + C.$$

The version of the substitution method for definite integrals is:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

subtended angle An angle is subtended by an *arc* (usually, of a circle) if the two rays forming the angle pass through the endpoints of the arc.

subtraction One of the four main arithmetic and algebraic operations. By subtracting two numbers we find their *difference*. Example: $-2-(-5)=3$. See also *addition and subtraction of complex numbers, fractions, functions, vectors, matrices.*

subtraction formulas for trigonometric functions See addition and subtraction formulas for trigonometric functions.

sum The result of *addition* of two or more numbers, functions, matrices, etc. Example: The sum of the functions $f(x)=2x^2+3x-1$ and $g(x)=x^2+5$ is the function $(f+g)(x)=3x^2+3x+4$. For the sum of infinitely many numbers or functions see *series* and *power series.*

summation notation A shorthand notation for summation of many numbers or functions. Uses the Greek letter sigma Σ. The notation

$$\sum_{n=1}^{10} n^2$$

means that we are adding the squares of all numbers from 1 to 10. The variable n is called *summation index* or variable. In case the index takes all natural numbers, the notation is $\sum_{n=1}^{\infty} a_n$.

sum rule See differentiation rules.

sum-to-product formulas In trigonometry, the formulas

$$\sin u + \sin v = 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\sin u - \sin v = 2\cos\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right),$$

$$\cos u + \cos v = 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right),$$

$$\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right).$$

Similar formulas are also possible for other functions but hardly ever used.

supplementary angles Two angles that add up to 180° (or π in radian measure). Two supplementary angles form one *straight angle.*

surfaces In the simplest cases a surface is a two dimensional object in three dimensional space that could be given by some equation $z=f(x,y)$. Depending on properties of the function f we get different type of surfaces. For example, if $f(x,y)$ has continuous partial derivatives by both variables then the surface would be called smooth. If the function is a second degree polynomial with respect to its variables then the surface would be called quadric. The meaning of the term closed surface should be understandable by itself. For explanations of terms *oriented surface, paramatric surface* see the corresponding entries.
For functions of three variables the equation $f(x,y,z)=k$ for some constant k is called the level surface.

surface area Suppose some surface S is given by the equation $z=f(x,y)$ and the function has continuous partial derivatives $f_x,\ f_y$ in some domain D. Then the area of the surface S could be calculated by the formula

$$A(S)=\int\int_D \sqrt{f_x^2(x,y)+f_y^2(x,y)+1}\,dA,$$

where $dA=dxdy$ is the element of the area of D. This formula is the three dimensional generalization of the *arc-length formula* and has similar format.

surface integral Suppose a surface S in three dimensional space is given by the function $z=g(x,y)$ over the domain D in xy-plane. The integral of the function $f(x,y,z)$ over that surface (under certain regularity conditions for that surface) is defined to be

$$\int\int_S f(x,y,z)dS =$$

$$\int\int_D f(x,y,g(x,y))\sqrt{g_x^2+g_y^2+1}dA.$$

Here $dA = dxdy$ is the element of the plane measure and dS is the element of the measure on the surface S and $g_x,\ g_y$ are the partial derivatives of the function $g(x,y)$. In particular case when $f(x,y,z) = 1$ we will get the *surface area* of S (see above). Similar formulas are valid for surfaces given by parametric equations and for integrals of vector functions with appropriate changes. For calculations of surface integrals of vector functions with the help of volume (triple) integrals see *divergence theorem*.

surface of revolution A surface received by rotation of some curve about an axis, such as coordinate axis. Many common surfaces (hyperboloids, paraboloids, etc.) could be viewed as surfaces of revolution.

symmetric function The term may mean different things depending on situation. The most commonly symmetry with respect to one of the coordinate axis or the origin is considered. Functions symmetric with respect to the y-axis are the even functions and functions symmetric with respect to the origin are the odd functions.

symmetric matrices A matrix A is symmetric if it is equal to its *transpose*: $A = A^T$. A symmetric matrix is necessarily a square matrix. Example:

$$\begin{pmatrix} 1 & -1 & 2 \\ -1 & -2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

synthetic division A simplified algorithm of division of any polynomial by a binomial of the form $x - c$. This method allows to avoid writing (and repeating) the variable x in the process of division. In order to be able to use this algorithm it is necessary to write the polynomial we are dividing in the form of decreasing powers and it is also mandatory to "fill-in" the missing powers. For example, the polynomial $P(x) = 2x^3 + 3x^2 - 4$ should be written as $P(x) = 2x^3 + 3x^2 + 0x - 4$. Now, suppose we want to divide this polynomial by $x + 1$. The synthetic division procedure looks as below

```
    |    2     3     0     -4
    |
 -1 |         -2    -1      1
----|----------------------------
    |    2     1    -1     -3
    |(result coefficients)(remainder)
```

The bottom row means that the result of the division is the polynomial $2x^2 + x - 1$ and the remainder is -3:

$$\frac{2x^3 + 3x^2 - 4}{x+1} = 2x^2 + x - 1 - \frac{3}{x+1}.$$

systems of linear algebraic equations The general system of m linear equations with n unknowns is

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

where $a_{ij},\ 1 \le i \le m,\ 1 \le j \le n$ and $b_k,\ 1 \le k \le n$ are numeric constants and $x_1, x_2, \cdots, x_n$ are the unknowns. For different methods of solving these kind of equations see *Cramer's rule, Gaussian elimination, Gauss-Jordan elimination, matrix method.*

systems of linear differential equations (homogeneous) This type of systems are written usually in the following form

$$x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x_n = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n$$

where $x_1, x_2, \cdots, x_n$ are the unknown functions. Using vector and matrix notations, this system could be written more compactly as $\mathbf{x}' = \mathbf{A}\mathbf{x}$. The solution of this system depends essentially on the *eigenvalues and eigenvectors* of the matrix A.

(1) The matrix A has n distinct real eigenvalues

$r_1, r_2, \cdots, r_n$ with corresponding (distinct) eigenvectors $(\xi_{\mathbf{1}}, \xi_{\mathbf{2}}, \cdots, \xi_{\mathbf{n}})$. Then the general solution vector of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 \xi_1 e^{r_1 t} + c_1 \xi_2 e^{r_2 t} + \cdots + c_1 \xi_n e^{r_n t}.$$

Example: To solve the system

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

we find the eigenvalues $r_1 = 3, \ r_2 = -1$ and the corresponding eigenvectors

$$\xi^{(\mathbf{1})} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \xi^{(\mathbf{2})} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

and the two (linearly independent solutions would be

$$\mathbf{x}^{(\mathbf{1})}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \quad \mathbf{x}^{(\mathbf{2})} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

(2) If the (real) matrix A has a complex eigenvalue then it should have also its complex conjugate eigenvalue: $r_1 = \lambda + i\mu, r_2 = \lambda - i\mu$. The corresponding eigenvectors are also conjugate in the sense that if $\xi^{(\mathbf{1})} = \mathbf{a} + \mathbf{ib}$ then $\xi^{(\mathbf{2})} = \mathbf{a} - \mathbf{ib}$. Now, the vector solutions corresponding to these complex eigenvalues could be written in the following form:

$$\mathbf{u}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t - \mathbf{b}\sin\mu t),$$

$$\mathbf{v}(t) = e^{\lambda t}(\mathbf{a}\cos\mu t + \mathbf{b}\sin\mu t).$$

Example: The system of equations

$$\mathbf{x}' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \mathbf{x}$$

has eigenvalues $r_1 = -1/2 + i, r_2 = -1/2 - i$. The two linearly independent vector solutions are

$$\mathbf{u}(t) = e^{-\frac{t}{2}} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \ \mathbf{v}(t) = e^{-\frac{t}{2}} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

(3) If any of the eigenvalues of the matrix A is repeated then the situation is more complicated. Example: For the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

$r = 2$ is a repeated eigenvalue and

$$\mathbf{x}^{(\mathbf{1})}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$$

is its corresponding vector solution. The second linearly independent vector solution is given by the formula

$$\mathbf{x}^{(\mathbf{2})}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}.$$

systems of linear differential equations (non-homogeneous) These systems in matrix form could be written as

$$\mathbf{x}' = \mathbf{a}\mathbf{x} + \mathbf{g}(t),$$

where $\mathbf{g}(t)$ is a vector function. The general solution of this system can be found by adding the general solution of the homogeneous system to any particular solution of the non-homogeneous system. Among many methods for finding particular solutions are the methods of *undetermined coefficients, variation of parameters, Laplace transform method* and the method of diagonalization. The first three methods (although for equations, not for systems) are described in separate articles. Here we will describe the diaginalizathion method.

If the matrix A is *diagonalizable*, then there is an invertible matrix T whose columns are the eigenvectors of A that diagonalizes A. Denoting $\mathbf{x} = T\mathbf{y}$ we will get an equation in new variable

$$T\mathbf{y}' = AT\mathbf{y} + \mathbf{g}(t).$$

After applying the inverse of T from the left of this equation we arrive to new equation

$$\mathbf{y}' = D\mathbf{y} + T^{-1}\mathbf{g}(t),$$

where $D = T^{-1}AT$ is a diagonal matrix. The last system is in fact a system of n equations where each of them contains only one variable and has the form

$$y_k'(t) = r_k y_k(t) + h_k(t), \ 1 \le k \le n.$$

Here r_k are in fact the eigenvalues if the matrix A. Each of these equations is a simple first order equation and can be solved by the method of *integrating* *factors*, for example. After solving it we just need to multiply the solutions by the matrix T to find the vector $\mathbf{x}$ and that will be a particular solution of the non-homogeneous system.

systems of non-linear algebraic equations There is no theory for solutions of non-linear algebraic systems but in many cases the substitution or elimination methods (but not *Gaussian elimination*) work here too. Example: Solve the system of equations

$$3x - y = 1$$
$$x^2 - y = 5$$

Solving the first equation for y and substituting into the second equation results in quadratic equation $x^2 - 3x - 4 = 0$. This equation has the solutions $x = 4, -1$. Substituting these values into either of the original equations gives $y = 11, -4$. Finally, the system has two solutions $(4, 11)$, $(-1, -4)$.

T

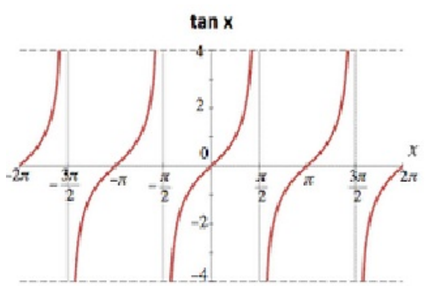
t-distribution Suppose the *random variable* x represents some distribution (normal or not) with mean μ and standard deviation σ and denote by $\bar{x}$ the means of samples of size n from that distribution. A new distribution formed by the values

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}},$$

where s is the sample standard deviation, is called t-distribution. This distribution is generally speaking not normal even if the original distribution of the x values is normal because of the presence of the *standard error* $SE(\bar{x}) = s/\sqrt{n}$ in denominator. The t-distribution is different for different values of sample size n and it is approaching *standard normal distribution* as the sample size increases. The values (probabilities) corresponding to t-distribution are found from special tables, by graphing calculators, or computers. It is generally acceptable to approximate t-distributions by the standard normal distribution if $n \geq 30$. The value t in the formula above is called one-sample t-statistic.

tangent function One of the six trigonometric functions. Geometrically, the tangent of an angle in a right triangle, is the ratio of the *opposite side* of the triangle to the *adjacent side*. Also could be defined as the reciprocal of the cotangent function. The function $\tan x$ could be extended as a function of all real numbers with exception of $x = \pi/2 + \pi n$, n any integer, and the range is all of R. $\tan x$ function is π-periodic.
Tangent function is related to other trigonometric functions by various identities. The most important of these are the identities $\tan x = \sin x / \cos x$, $\tan x = 1/\cot x$ and a version of the Pythagorean identity $1 + \tan^2 x = \sec^2 x$. The derivative and integral of this function are given by the formulas

$$\frac{d}{dx} \tan x = sec^2 x, \int \tan x dx = \ln |\sec x| + C.$$



tangential component Any vector coming out from some point of a curve, can be presented as a combination of *tangent and normal vectors*. The component in the direction of the tangent vector is the tangential component.

tangent line A line is tangent to some curve at a point of that curve, if the line and curve coincide at that point and have no other common points in some neighborhood of that point. If the curve is given by a *differentiable function* $f(x)$, then the tangent line to this curve at the point $(a, f(a)$ has the slope

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

and could be given by the equation $y - f(a) = f'(a)(x - a)$. If the function is not differentiable at some point then the corresponding curve does not have a tangent line at that point.

tangent line method The same as the Euler method for approximate solutions of differential equations.

tangent plane A plane in three dimensional space is tangent to some surface at some point of that surface if they coincide at that point but have no other common points in some neighborhood of that point. If the surface is given by a differentiable function $z = f(x, y)$, then the tangent plane at the point (x_0, y_0, z_0) is given by the equation

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

tangent vector A *vector* (usually unit vector) in the direction of the tangent to a curve at some point. Tangent vector is *orthogonal* to the *normal vector* at the same point. If a space curve is given by the parametric vector function $\mathbf{r}(t)$, then its derivative $\mathbf{r}'(t_0)$, if it exists and is not zero, is the tangent vector at the point $\mathbf{r}(t_0)$.

tautochrone This is a curve down which a particle slides freely under just the force of gravity and reaches the bottom of the curve in the same amount of time no matter where its starting point on the curve was. It turns out that the only curve satisfying this condition is the cycloid.

Taylor series Let the function $f(x)$ be defined on some interval around the point $x = a$ and has derivatives of arbitrary order at that point. Then the *power series*

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a)$$

$$+\frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

is called the Taylor series of f. The Taylor series of a function may or may not converge to represent the function itself. The functions such that their Taylor series converges to them are called *analytic*.
A finite part of the Taylor series with summation from $n = 0$ to $n = N$ is called the Taylor polynomial of order N. The Nth Taylor polynomial is usually denoted by $T_N(x)$. If we denote the remaining part of the Taylor series (the sum from $n = N + 1$) by $R_N(x)$, then the question of when the Taylor series converges to the function is resolved by the following Theorem. If $f(x) = T_N(x) + R_N(x)$ and

$$\lim_{N\to\infty} R_N(x) = 0$$

for $|x-a| < R$, then f is equal to the sum if its Taylor series on $|x - a| < R$.

A criteria showing when the remainder of the Taylor series converges to zero is given by the following Taylor's inequality:

If $|f^{(N+1)}(x)| \leq M$ for $|x-a| \leq r$, then the remainder of the Taylor series satisfies

$$|R_N(x) \leq \frac{M}{(N+1)!}|x-a|^{N+1}, \quad |x-a| \leq d.$$

The special case of Taylor series when $a = 0$ is called MacLaurin series. See corresponding entry for examples of Taylor-MacLaurin series.

Taylor's inequality See *Taylor series.*

term of a sequence In a *sequence* $a_1, a_2, a_3, \cdots$ each member (element, entry) is called term. Similarly, in the series $\sum_{n=1}^{\infty} a_n$ each a_n is a term of the series. The same term(word) is used to describe the elements of any polynomial, which also are called monomials.

terminal side Of two sides forming an angle in standard position, one is called the initial side and the other is the terminal. See the entry angle for all the details.

term-by-term differentiation and integration If a power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$ then the function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable on the interval $(a - R, a + R)$ and

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

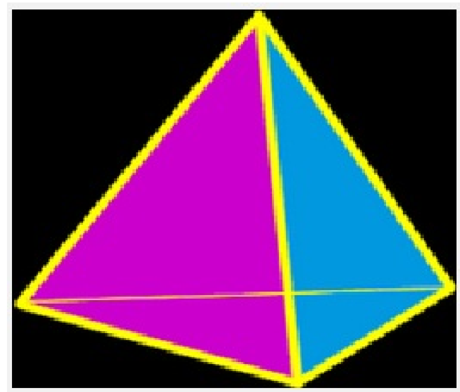
$$\int f(x)dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}.$$

The differentiated and integrated series have the same radius of convergence as the original series.

tests of convergence of series Different tests to establish convergence or divergence of *numeric or power series.* Each of them have separate entries. See *alternating series convergence test, comparison test, integral test, limit comparison test, ratio test, root test.*

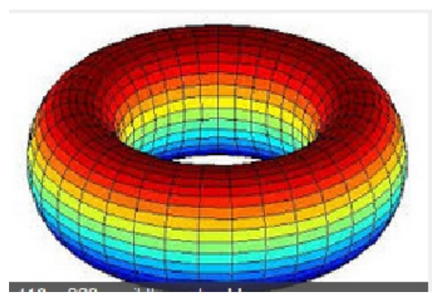
test of significance Statistical test, establishing the significance of the sample results. As a rule, this means that some test statistic should be compared to some pre-determined small value (called α-value). The finding considered significant if the test statistic (usually *P-value*) is smaller than this α-value.

tetrahedron A geometric object composed of four triangular faces, three of which meet at each vertex. A regular tetrahedron is one in which the four triangles are regular, or equilateral.



time plot A type of graph, mostly used in Statistics, where the points are found from observations taken during some time period and then connected by segments of straight line.

torus A geometric three dimensional surface that is the result of rotation of a circle around an axis not crossing or touching that circle.



total differential For a function $z = f(x, y)$ of two variables, the expression

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Total differential for functions of three or more variables is defined in the similar way.

trace of a matrix For a *square matrix* A the trace is the sum of all elements on the main diagonal. For the matrix

$$\begin{pmatrix} 2 & -2 & 0 \\ 5 & -1 & -4 \\ 3 & 0 & 4 \end{pmatrix}$$

the trace is $2 + (-1) + 4 = 5$.

transcendental function Any function that is not algebraic. Transcendental functions include, but are not limited to, exponential, logarithmic, trigonometric, hyperbolic, and inverse trigonometric and hyperbolic functions. Besides, most of the functions defined with the use of power series or integrals are also transcendental.

transcendental number A number that is not a solution of an *algebraic equation* with rational coefficients. Examples are the numbers e and π.

transformation See linear transformation.

transformations of a function The combined name for operations on functions including *horizontal shift, vertical shift, reflection, stretching, shrinking (or compressing).* The most general transformation of a given function $f(x)$ would be the function $g(x) = af(bx - c) + d$. Here the constant b indicates the horizontal stretching (if $|b| < 1$) or shrinking (if $|b| > 1$). The number c/b indicates the horizontal shift, d is the vertical shift and the constant a is the vertical stretching (if $|a| > 1$) or shrinking (if $|a| < 1$). Also, multiplication by the constant a may result in reflection with respect to the x-axis if it is negative. Similarly, if $b < 0$ then the graph of the function f would be reflected with respect to the y-axis. A typical example of a function with all of the above transformations is the function

$$y = -2\sin(3x + \pi/4) - 1.$$

The term translations of functions is also sometimes used.

transition matrix Suppose V is a vector space and $B = \{\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_n}\}$ and $B' = \{\mathbf{u'_1}, \mathbf{u'_2}, \cdots, \mathbf{u'_n}\}$ are two bases in that space. Let for some vector $v \in V$ $[\mathbf{v}]_\mathbf{B}$ and $[\mathbf{v}]_{\mathbf{B'}}$ denote the *coordinate matrices* of that vector with respect to the bases B and B' respectively. Then there exists a $n \times n$ matrix P such that $[\mathbf{v}]_B = P[\mathbf{v}]_{B'}$. The matrix P is the transition matrix from the basis B to the basis B' and could be written in the form

$$P = \left[[\mathbf{u'_1}]_B | [\mathbf{u'_2}]_B | \cdots | [\mathbf{u'_n}]_B\right].$$

transpose of a matrix The transpose of a matrix A is a matrix such that the rows are the columns of A and the columns are the rows of A. Hence, if A is a $m \times n$ matrix, then its transpose is a $n \times m$ matrix. The most common notation for the transpose is A^T. Example: For the matrix

$$\begin{pmatrix} 1 & -2 & 0 & -2 \\ 5 & 1 & -4 & 3 \\ 3 & 0 & 1 & 2 \end{pmatrix}$$

the transpose is

$$\begin{pmatrix} 1 & 5 & 3 \\ -2 & 1 & 0 \\ 0 & -4 & 1 \\ -2 & 3 & 2 \end{pmatrix}.$$

transposition In any set of elements the operation that switches any two elements but leaves all the others unchanged is called a transposition. It is a special type of permutation.

transverse axis of hyperbola The axis (line) passing through two *foci* of the *hyperbola*.

trapezoid A *quadrilateral* with one pair of parallel sides.

trapezoidal rule One of the methods of *approximate integration*. To calculate the approximate value of the integral $\int_a^b f(x)dx$ using trapezoidal rule we divide the interval $[a, b]$ into n equal intervals of the length $\Delta x = (b - a)/n$ with the points $x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \cdots, x_n = b$. Then

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

For the degree of accuracy of this approximation see error estimate for the trapezoidal rule.

triangle A geometric figure that consists of three points (vertices) and three intervals connecting these points. Any two adjacent intervals form an angle, hence the name. Triangle is the simplest special case of a *polygon*.

triangle inequality For any two vectors $\mathbf{u}$ and $\mathbf{v}$ in the Euclidean space R^n

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||,$$

where the symbol $||\cdot||$ indicates the *norm (length, magnitude)* of a vector. The name comes from the fact that this inequality generalizes the geometric fact that any side of a triangle is shorter than the sum of the other two sides. The triangle inequality has further generalizations to the so-called metric spaces.

triangular matrix A matrix where all the elements above or below the main diagonal are zeros. See also *lower triangular, upper triangular* matrices.

trigonometric equations *Equations* involving trigonometric functions. Unlike identities, equations are not true statements for all or "almost all" values of the variable, but might be true only for limited number of values if considered in finite interval. In cases when we are looking for solutions without restrictions on the interval, we usually get infinitely many solutions but they are always periodic repetitions of solutions on the finite interval. Examples: The equation

$$\sin x = 1/2$$

has only two solutions $x = \pi/6, x = 5\pi/6$ on the interval $[0, 2\pi)$. If considered on the real axis $(-\infty, \infty)$, if has infinitely many solutions found by adding arbitrary multiple of 2π to the two basic solutions: $x = \pi/6 + 2\pi k,\ x = 5\pi/6 + 2\pi k,\ k = 0, \pm1, \pm2, \cdots$. Similarly, the equation

$$4\cos^2 x - 1 = 0$$

has the solutions $x = \pi/3,\ x = 2\pi/3$ on $[0, 2\pi)$ and we get the general solution by adding a multiple of 2π to that solutions.

trigonometric form of the complex number Let $z = a + ib$ be a complex number. Then it can be represented on the *complex plane* as a point with coordinates a and b. The distance of this point from the origin is the *modulus* of the number $r = \sqrt{a^2 + b^2}$. Now, if we connect the origin with the point z, that segment will form some angle θ with the *real axis*, called the argument of z. The argument of a number can be found from the formula $\tan\theta = b/a$.The trigonometric form of the number z is

$$z = r(\cos\theta + \sin\theta).$$

The Euler modification of this form is $z = re^{i\theta}$. Trigonometric form is convenient when multiplying, dividing, raising to whole power and especially for finding the roots of complex numbers. See multiplication of complex numbers, division of complex numbers, DeMoivre's theorem,roots of a complex number.

trigonometric functions The six functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$ and $\csc x$. For the definitions and graphs of each of them see the corresponding definitions.

trigonometric identities Identities involving trigonometric functions. Identities can be true for all values of the variable(s) or for all values except where the functions are not defined. Trigonometric formulas are also identities. Examples:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos x + \sin x \tan x = \sec x$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y.$$

trigonometric integrals A group of indefinite integrals that contain different combinations of trigonometric functions. For most of these combinations there are methods to evaluate that integrals but not all trigonometric integrals can be expressed in the form of elementary functions.
1) Integrals of the form $\int \sin^n x \cos^m x dx$.
(a) If $m = 2k + 1$ is odd, then write

$$\int \sin^n x (\cos^2 x)^k \cos x dx$$

$$= \int \sin^n x (1 - \sin^2 x)^k \cos x dx$$

and using the substitution $u = \sin x$ reduce the integral to algebraic: $\int u^n (1 - u^2)^k dx$.

(b) If $n = 2s + 1$ is odd, then similar to the previous case

$$\int (\sin^2)^s \cos^m x \sin x dx$$

$$= \int (1 - \cos^2 x)^s \cos^m x \sin x dx$$

and the substitution $u = \cos x$ again reduces integral to algebraic.

(c) If both n and m are even, then the use of *half-angle formulas*

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \ \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

(possibly multiple times) reduces the integral to easy to evaluate form.

2) Integrals of the form $\int \tan^n x \sec^m x dx$.

(a) If $m = 2k$ is even, then write

$$\int \tan^n x (\sec^2)^{k-1} \sec^2 x dx$$

$$= \int \tan^n x (1 + \tan^2 x)^{k-1} \sec^2 x dx$$

and substitute $u = \tan x$ using the fact that $(\tan x)' = \sec^2 x$. Integral reduces to algebraic.

(b) If $n = 2s + 1$ is odd, then write

$$\int (\tan^2 x)^s \sec^{m-1} x \sec x \tan x dx$$

$$= \int (\sec^2 x - 1)^s \sec^{m-1} \sec x \tan x dx$$

and substitute $u = \sec x$ using the fact that $(\sec x)' = \sec x \tan x$. Once again, the integral reduces to algebraic.

3) In evaluating the integrals of the form $\int \sin nx \cos mx dx, \int \sin nx \sin mx dx$ or $\int \cos nx \cos mx dx$ it is useful to use the *product-to-sum* formulas.

Many other trigonometric integrals can be calculated by different methods. Among trigonometric integrals that cannot be expressed with the use of elementary functions is the integral $\int \sin(x^2) dx$.

Examples:

$$\int \sin^5 x \cos^2 x dx = \int (\sin^2 x)^2 \cos^2 x \sin x dx$$

$$= \int (1 - \cos^2 x)^2 \cos^2 x \sin x dx$$

$$= -\int (1 - u^2)^2 u^2 du = -\frac{u^3}{3} + 2\frac{u^5}{5} - \frac{u^7}{7} + C$$

$$= -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C.$$

$$\int \tan^6 x \sec^4 x dx$$

$$= \int \tan^6 x (1 + \tan^2 x) \sec^2 x dx$$

$$= \int u^6 (1 + u^2) du = \frac{u^7}{7} + \frac{u^9}{9} + C$$

$$= \frac{\tan^7 x}{7} + \frac{\tan^9}{9} + C.$$

$$\int \sin 7x \cos 4x dx$$

$$= \frac{1}{2}\left[\int \sin 3x dx + \int \cos 11x dx\right]$$

$$== -\frac{1}{6}\cos 3x + \frac{1}{22}\sin 11x + C.$$

trigonometric series A finite or infinite series of the form

$$c + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta).$$

In the case the coefficients c, a_n, b_n have special relations to some function $f(x)$, the series becomes a Fourier series. See the corresponding entry.

trigonometric substitutions See integration by trigonometric substitution.

trinomial A *polynomial* that has three terms. Trinomial might have one or more variables. Examples: $2x^2 - 5x + 6, \ 3x^3y^2 + 4x^2y^4 - 6xy$.

triple product Or scalar triple product of three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in three dimensional space is defined as $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ and could be calculated by the formula

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

trivial solution An obvious, very simple solution of an equation or system of equations. As a rule, trivial solution is a zero solution. Any homogeneous linear system has a trivial solution. Many differential equations have trivial (identically zero) solution along with non-zero solutions.

twisted cubic A space curve given by the vector-function $\mathbf{r}(t) = (t, t^2, t^3)$.

U

unbiased statistic A sample statistic whose *sampling distribution* has a mean value equal to the mean value of the *population parameter* that it is estimating. Sample mean, proportion and variance are unbiased statistics but the sample standard deviation is not.

unbounded function A function that is not bounded from above, below, or both. The functions $f(x) = \ln x, g(x) = x^2$ are unbounded but the function $h(x) = \sin x$ is bounded.

unbounded region In case of a plane *region*, it is called bounded if it could be enclosed in some circle of finite radius centered at the origin. If there is no such circle, then the region is unbounded. Any region given by an inequality $y > ax + b$ is an example of an unbounded region. Unbounded regions in the three dimensional space are defined similarly.

undefined function A function that is not or cannot be defined for certain values of the variable. For example, the function $f(x) = 1/x$ is undefined for $x = 0$ because division by zero is not defined.

undercoverage (of population) In statistics, when sampling, sometimes certain parts of the population are covered less that the others. For example, when conducting telephone polls, people without phones are not covered at all.

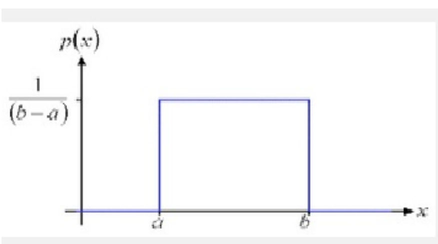
underdetermined linear system A system of linear equations that has more unknowns than equations. If this system is consistent, then it has infinitely many solutions. See also *overdetermined linear systems.*

undetermined coefficients Also called the method of undetermined coefficients for certain types of equations. In using this method we either know or guess the form of the solution but do not know the value of the constant(s) and are trying to determine that value. One of the versions of

this method is described in the definition of partial fraction decomposition. Second application is for solution of non-homogeneous linear differential equations, working mainly for equations with constant coefficients.
Suppose we are trying to find a *particular solution* of the equation

$$y'' - 3y' - 4y = 3e^{2t}.$$

We guess that the solution should have the form $Y(t) = Ae^{2t}$, where A is yet to be determined coefficient. After we plug-in this supposed solution into the equation we get $(4A - 6A - 4A)e^{2t} = 3e^{2t}$ and $A = -1/2$ and the coefficient is found.
Similar method also works for the systems of linear equations where, of course, we have to solve systems of linear algebraic equations with respect to a number of undetermined coefficients.



uniform distribution A continuous *probability distribution* where the *density curve* is a function that is constant on some interval $[a, b]$ and is zero everywhere else. Because of the requirement that the area under the curve should be 1, the height of the line is $1/(b - a)$.

union of sets If A and B are two sets of some objects, then their union $A \cup B$ is defined to be the set of all elements that belong to A or B or both of them. Example: If $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$ then $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$. In case when the two sets have common elements, then in the union these elements are counted only once. Hence, if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, then $A \cup B = \{1, 2, 3, 4\}$.

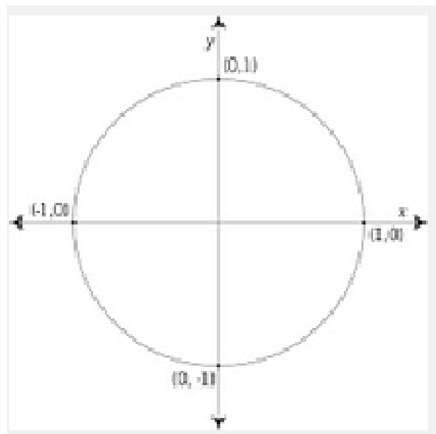
uniqueness theorems See existence and uniqueness theorems.

unit step function Similar to the *Heaviside function*, defined by the formula

$$u_c(t) = \begin{cases} 1 & \text{if} \quad t \geq c \\ 0 & \text{if} \quad t < c \end{cases}$$

for $c \geq 0$. The *Laplace transform* of this function is the function $\mathcal{L}[u_c(t)] = e^{-cs}/s, \ s > 0$.

unit circle A circle with radius equal to one. In *Cartesian* coordinate system such a circle could be given by the equation $(x - a)^2 + (y - b)^2 = 1$, where (a, b) is the coordinate of the center. The picture shows the unit circle with center at the origin.



unit sphere A sphere with radius one. In *Cartesian* coordinate system the equation is $x^2 + y^2 + z^2 = 1$. The notion of the sphere extends also to higher dimensional Euclidean spaces.

unit vector A vector that has *norm (length, magnitude)* equal to 1. If $\mathbf{v}$ is an arbitrary non-zero vector in the vector space V, then the vector $\mathbf{u} = \mathbf{v}/||\mathbf{v}||$ has unit norm and hence, is a unit vector. See also *normal vector, tangent vector.*

unitary diagonalization Suppose A is some *square matrix* and P is a *unitary matrix* of the same size. Then A is said to be unitary diagonalizable if the matrix $P^{-1}AP$ is *diagonal.*

unitary matrix A matrix A with complex entries is unitary, if its *inverse* A^{-1} is equal to its *conjugate transpose* A^*. The matrix

$$\frac{1}{2}\begin{pmatrix} 1+i & 1+i \\ 1-i & 1-i \end{pmatrix}.$$

is a unitary matrix.

unlike fractions Two (or more) fractions that have different denominators. The fractions $\frac{3}{4}$ and $\frac{4}{7}$ are unlike fractions. On the other hand, the fractions $\frac{3}{8}$ and $\frac{6}{16}$ are not unlike fractions because the second one could be reduced to simplest terms as $\frac{3}{8}$ with the same denominator.

unlike terms Two terms in an algebraic expression that do not have identical variable parts (factors). In the expression $2xy + 2xy^2$ the two terms are unlike terms, because xy and xy^2 are not identical. The constant factors do not play role in deciding if the terms are *like terms* or unlike terms.

upper triangular matrix A *square matrix* where all the entries below the *main diagonal* are zeros. Example:

$$\begin{pmatrix} 2 & 4 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{pmatrix}.$$

The determinant of such a matrix is just the product of all diagonal elements. See also *lower diagonal matrix.*

V

value of a function The numeric expression of a given function at a given point. Examples: (1) The value of the function $f(x) = 3x^3 - 2x^2 + 5x - 1$ at the point $x = 2$ is determined by substituting the value $x = 2$ in the expression of the function: $f(2) = 3 \cdot 2^3 - 2 \cdot 2^2 + 5 \cdot 2 - 1 = 25$. (2) The value of the function $f(x) = \sin x$ at the point $x = \pi/6$ is $1/2$.
To calculate the values of more complicated functions approximate methods and/or calculators are needed.

variable One of the basic mathematical objects along with constants, functions, etc. Variables are quantities that do not have fixed value (as the constants) but rather have the ability to change. For example, time is a variable quantity because it does not stand still. Variables are denoted by symbols such as x, y, z, t, u, and others. In certain situations there might be two or more variable quantities and if they are related to each other by some rule or formula, we call that a *function or relation.* Accordingly, the variables in that relations would be *independent or dependent.* See also the entries *explanatory, quantitative, qualitative, response, random, lurking variables.*

variance For a set of numeric values $x_1, x_2, x_3, \cdots, x_n$ with the mean μ, the variance is defined by the formula

$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n-1}.$$

The square root of this quantity is called *standard deviation.*

variation Specific types of relationships between two or more variable.
(1) Direct variation. In this case one of the variables is directly proportional to another variable in some form. For example if $y = kx$ (k is some numeric constant) we say that y varies directly with x. If $y = kx^2$ then y varies directly with x^2.
(2) Inverse variation. In this case one variable is inversely proportional to some form of the other vari-

able. If $y = k/x$ we say that y varies inversely with x. If $y = k/\sqrt{x}$, then y varies inversely with $\sqrt{x}$, and so on.
(3) Joint variation. In this case one variable is directly or inversely proportional to some forms of two or more other variable. Example: $z = kx^2/y^3$.

variation of parameters A method of finding a particular solution for *non-homogeneous linear differential equations*. This method is the development of the method of undetermined coefficients and allows to find particular solutions for much wider classes of equations.
Suppose we want to find the general solution of the equation

$$y'' + p(t)y' + q(t)y = g(t). \qquad (1)$$

First of all, the general solution of this equation is the combination of the general solution of the corresponding *homogeneous equation*

$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$

and any particular solution of the equation (1). Assume also that we know the general solution of the homogeneous equation (2) and it is

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t),$$

where $y_1,\ y_2$ are two *linearly independent* solutions. The idea behind the method of variation of parameters is to replace constants $c_1,\ c_2$ by some, yet to be determined functions $u_1(t),\ u_2(t)$ and try to find a particular solution in the form

$$Y(t) = u_1(t)y_1(t) + u_2(t)y_2(t).$$

Theorem. The particular solution of the equation (1) could be written in the form

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}ds$$

$$+y_2(t)\int_{t_0}^{t} \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}ds,$$

where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2.
Example: To solve the equation

$$y'' + 4y = 3\csc t$$

we notice that the corresponding homogeneous equation $y'' + 4y = 0$ has the general solution $y(t) = c_1\cos 2t + c_2 \sin 2t$ and look for a particular solution of the non-homogeneous equation in the form $Y(t) = u_1(t)\cos 2t + u_2(t)\sin 2t$. Now, the Wronskian of this equation is $W = 2$ and by the Theorem above, $u_1(t) = -3\cos t$ and $u_2(t) = 3/2 \ln|\csc t - \cot t| + 3\cos t$. Plugging in these functions into the previous formula we find a particular solution of the non-homogeneous equation:

$$Y(t) = -3\sin t\cos 2t + 3\cos t \sin 2t$$

$$+\frac{3}{2}\sin 2t \ln|\csc t - \cot t|.$$

Similar method works also for equations of higher order and for systems of non-homogeneous equations.

vector One of the basic mathematical objects, vectors could be viewed as quantities that have both some numeric value and a direction. Many physical notions (such as force or pressure) have that properties and vectors are abstract mathematical representations of these facts. As with the other mathematical objects (numbers, variables, functions, matrices), many operations are possible to perform with vectors too. As objects, vectors may exist on the plane, three dimensional space, or more generally, in any vector space.
1) Addition of vectors in the plane is defined by the *parallelogram rule* (see *addition and subtraction of vectors*). More generally, if $\mathbf{v}$ and $\mathbf{u}$ are two vectors in some vector space V, then

$$\mathbf{v} + \mathbf{u} = (v_1 + u_1, v_2 + u_2, \cdots, v_n + u_n),$$

where $\mathbf{v} = (v_1, v_2, \cdots, v_n),\ \mathbf{u} = (u_1, u_2, \cdots, u_n)$.
2) The scalar product of a number c and the vector $\mathbf{v}$ is defined as $c\mathbf{v} = (cv_1, cv_2, \cdots, cv_n)$.
3) It is impossible to define a product of two vectors in any space V in such a manner that the result be another vector from the same space and at the same time satisfy all the expected properties of multiplication (commutative, associative, distributive). Instead, the *inner product* (or *dot product*) of two vectors could be defined which is not a vector but a number (real or complex, depending on the space

V): $\mathbf{v} \cdot \mathbf{u} = v_1u_1 + v_2u_2 + \cdots + v_nu_n$. See also *cross product* for one specific type of vector multiplication.
4) The length (or magnitude, or norm) of a vector is now defined as $||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.
5) The angle between two vectors now could be defined with the help of inner product as follows:

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{u}}{||\mathbf{v}||||\mathbf{u}||}.$$

Two vectors are called parallel if the angle between them is either zero or $\pi(=180°)$. In case when the angle is $\pi/2 = 90°$, the vectors are called perpendicular, or orthogonal.
For additional facts about vectors see also *basis, linear dependence and independence of vectors, tangent vector, unit vector, zero vector.*

vector field Let D be a region on the plane. A vector field $\mathbf{F}$ is a function that maps any point (x, y) from D to another point on the plane. This vector field could be written now as

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j},$$

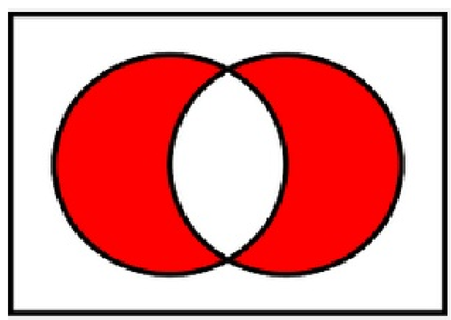
where P, Q are ordinary functions of two variables. Vector fields can be defined also for three or any higher number of variables in exactly the same way. See also *curl of the vector field, flux, gradient vector field.*

vector function A vector such that the coordinates (components) are functions. Example: $\mathbf{F}(t) = (f(t), g(t), q(t)) = (\sin t, \cos t, t)$.

vector product The product of two vectors is impossible to define in a way that the result be another vector from the same space with expected properties of multiplication as we have them for numbers, variables, or functions. For special forms of vector products see cross product, inner product.

vector space The abstract vector space V is a collection of objects called *vectors* for which the operations of *vector addition* and *scalar multiplication* are defined. Denoting the vectors by $\mathbf{u}, \mathbf{v}, \mathbf{w}, ...$ and scalars by $k, \ell, ...$ we require that they satisfy the following list of *axioms* of the vector space:
(1) If $\mathbf{u}$ and $\mathbf{v}$ belong to V, then $\mathbf{u} + \mathbf{v}$ also belongs to V;
(2) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;
(3) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$;
(4) There is a vector $\mathbf{0}$ in V, named zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$;
(5) For each $\mathbf{u}$ in V there is a vector $-\mathbf{u}$ also in V, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$;
(6) If $\mathbf{u}$ is in V then $k\mathbf{u}$ is also in V;
(7) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$;
(8) $(k + \ell)\mathbf{u} = k\mathbf{u} + \ell\mathbf{u}$;
(9) $k(\ell\mathbf{u}) = (k\ell)\mathbf{u}$;
(10) $1\mathbf{u} = \mathbf{u}$.
If the scalars $k, \ell, ...$ are real then the vector space is called real vector space. In case when we chose complex scalars, the space is called complex vector space. Among the most common examples of vector spaces are the *Euclidean space* R^n, the space of all $m \times n$ matrices, the space of all polynomials of degree n or less, and many others.

Venn diagram A graphical method of representing sets and their relations. Used in set theory, logic, statistics, and probability theory. Especially helpful in understanding the *union, intersection of sets.* Below is a typical Venn diagram of intersection of two sets.



vertex A term used in many different situations with slightly different meaning. In some sense vertices are the "extreme points" of geometric objects and figures. For example, the vertex of an angle is the point from where the two rays forming the angle come out (see also *angle*). For a polygon or a polytope the vertices are the "corner" points of that figures. At the same time the notion of the vertex also applies to other geometric figures, such as ellipses, parabolas, and hyperbolas. See the corresponding entries for exact descriptions.

vertical asymptote The (vertical) line $x = a$ is a vertical asymptote for a function $f(x)$ if that function grows unboundedly as the point x approaches the point a. In terms of limits this property is presented by one of the following limit conditions:

$$\lim_{x\to a} f(x) = \infty, \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a^+} f(x) = \infty, \qquad \lim_{x\to a} f(x) = -\infty$$

$$\lim_{x\to a^-} f(x) = -\infty, \qquad \lim_{x\to a^+} f(x) = -\infty.$$

These conditions mean that the function cannot have a finite (specific) value at that point. As a result, the function's graph can never cross the vertical asymptote. Examples: (1) For the function $f(x) = 1/x^2$ the line $x = 0$ (the y-axis) is the only vertical asymptote. (2) The function $f(x) = x^2/(x-1)(x+2)$ has vertical asymptotes $x = 1$ and $x = -2$.

vertical line test A geometric method of determining if a given graph is a graph of a function or a relation. If any vertical line crosses the graph only once, then it is a function, because that means that to each value of x there is only one corresponding value of y. If this condition is violated at even one point we have a relation instead of a function. This method is not precise because it depends on graphing skills that are subjective, still it is a useful visual tool.

vertical shifts of graphs Addition or subtraction of any constant to any given function results in the vertical shift (up or down) of the graph of that function. Example:

$$f(x) = \sin x, \qquad g(x) = \sin x + 2.$$

The graphs of f and g are identical except that the second one takes values two more than the first one. Its graph is two units above the graph of f.

volume A geometric notion indicating the amount of space occupied by an object in three-dimensional space. The volumes of simple geometric solids, such as *prisms, pyramids, spheres, cylinders, cones*, and others are known for a long time. For calculations of volumes of more complicated solids it is necessary to involve calculus. In general, if a solid is defined by a function $z = f(x, y)$, where the variables (x, y) are changing on some rectangle $D = [a, b] \times [c, d]$, then the volume of that solid is given by the *double integral*

$$V = \int_a^b \int_c^d f(x, y)\,dx\,dy.$$

In cases when the solid is a solid of revolution of some curve about some axis, or if the solid has some other extra properties of symmetry, the volume could be calculated by the method of cylindrical shells or discs (washers).

W

wave equation For differential equations. In case of a function of two spacial variables x, y, and the time variable t, the equation

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = \frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2}.$$

In the simpler case of one spacial variable x and time variable t the equation has the form $a^2 u_{xx} = u_{tt}$.

weighted mean Or weighted average. In cases when different numbers have different weights, their average or mean is calculated using the weighted mean formula. Let $w_1, w_2, \cdots, w_n$ be all positive numbers (some may be zero, but not all) and assume that $x_1, x_2, \cdots, x_n$ are numeric values with corresponding weights. This means that the value x_i has the weight w_i, $1 \le i \le n$. Then the weighted mean is

$$\overline{x}_w = \frac{w_1 x_1 + w_2 x_2 + \cdots + w_n x_n}{w_1 + w_2 + \cdots + w_n}.$$

In cases when the sum of w_i is 1, the formula simplifies to $\overline{x}_w = \sum_{i=1}^{n} w_i x_i$. Example: Assume a student get the scores $62, 78, 81, 92$ on her tests, 84 on the Final, and 90 for her homework. Each test is valued at 15%, the Final is 30% and homework is valued at 10%. Then the mean score of the student will be $0.15(62+78+81+92) + 0.3 \cdot 84 + 0.1 \cdot 90 = 72.95$, because the sum of all weights is one (0.15+0.15+0.15+0.15+0.3+0.1=1).

weighted inner product Let $\mathbf{u} = (u_1, u_2, \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$ be two vectors in the Euclidean space R^n and assume $w_1, w_2, \cdots, w_n$ is a sequence of positive numbers. Then the weighted inner product with this sequence (weight) is

$$\langle \mathbf{u}, \mathbf{v} \rangle_w = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n.$$

whole number The set of the whole numbers consists of all *natural* numbers with addition of the number zero. Hence, the whole numbers are: $0, 1, 2, 3, 4, \cdots$

Wronskian Consider the second order linear homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

with the initial value conditions $y(t_0) = y_0,\ y'(t_0) = y_0'$. Assume also that y_1 and y_2 are two solutions of the equation (without initial conditions). The determinant

$$\det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}$$

is called the Wronskian of the equation and plays important role in finding the general solution of the equation. See Abel's formula for the presentation of the Wronskian and variation of parameters for an application. Wronskian also generalizes to the case of homogeneous equations of arbitrary order n. For the equation

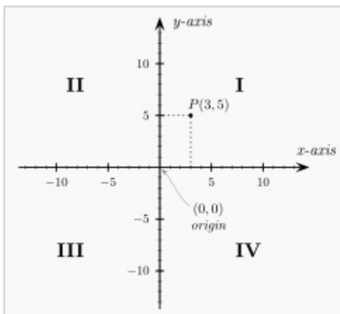
$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

with solutions $y_1, y_2, \cdots, y_n$, the Wronskian is the determinant

$$\det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

XYZ

x-axis In *Cartesian* coordinate system on the plane the horizontal axis. The projection of any point on the plane onto the x-axis gives the x*-coordinate* of the point, always written first. For example, for the point $(5, -3)$ the first entry 5 is the x-coordinate.



x-coordinate See above, x-axis.

x-intercept The intersection of the graph of any function with the x*-axis*. For polynomials, trigonometric, logarithmic, exponential and other functions, these points coincide with the *zeros* of the corresponding types of equations. Examples: (1) To find the x-intercepts of the function $f(x) = x^3 + 3x^2 - 4x - 12$ we solve the corresponding algebraic equation $x^3 + 3x^2 - 4x - 12 = 0$. This equation has the zeros $x = 2, -2, -3$ and that points are exactly the x-intercepts of the function $f(x)$; (2) The function $f(x) = \sin 2x$ has the intercepts $x = \pi n/2, n = 0, \pm 1, \pm 2, \cdots$ because they are the solutions of the corresponding trigonometric equation $\sin 2x = 0$.

y-axis In *Cartesian* coordinate system on the plane the vertical axis. The projection of any point on the plane onto the y-axis gives the y*-coordinate* of the point, always written second. For example, for the point $(5, -3)$ the second entry -3 is the y-coordinate. See the figure above.

y-coordinate See above, $y - axis$.

y-intercept The intersection of the graph of any function with the y*-axis*. If that function is defined for the value $x = 0$ then it always has the y-intercept. Unlike x-intercepts, a function may have no more than one y-intercept. On the other hand, a relation may have more than one y-intercept. Examples: (1) The function $f(x) = x^3 + 3x^2 - 4x - 12$ has the value $f(0) = -12$ and that is the only y-intercept of that function; (2) The function $f(x) = 1/x$ is not defined for $x = 0$ and, as a result, does not have a y-intercept; (3) The relation $x^2 + y^2 = 1$ representing the unit circle has two y-intercepts at the points $(0, 1)$ and $(0, -1)$.

z-axis In three-dimensional *Cartesian* coordinate system, in addition to x- and y-axes, there is also the z-axis, that is perpendicular to the plane formed by these two axes. Any point in the space has three orthogonal projections, corresponding to three axes. The projection onto the z-axis gives the z*-coordinate* of the point that is always written on the third position: For the point (1,2,3) the third entry 3 is the z-coordinate.

z-coordinate See above, z-axis.

z-score Suppose x is a random variable and assume $\bar{x}$ is its mean and s its standard deviation. The quantity $z = (x - \bar{x})/s$ is the z-score (or standardized value) of the variable x. If the variable x belongs to a normal distribution with the mean $\bar{x}$ and standard deviation s, then the variable z (the z-score) belongs to *standard normal distribution*: its mean is zero and standard deviation is 1. The z-score measures the distance of the value x from the mean in terms of standard deviation units. z-scores are also sometimes called z statistic.

zero Originally, one of the integers in the real number system that is smaller that 1 but greater than -1. The only real number that is neither positive nor negative. The most important property of zero is that it is the *additive identity* which is expressed algebraically as $a + 0 = a$.
The importance of the number zero goes beyond the real numbers because it serves as the additive identity also for the complex numbers: If z is any complex number, then $z + 0 = z$. Also, zero could be considered as the "zero function" that for all values of the variable x equals to zero. In this interpretation zero

is also the additive identity in the set of all functions. See also *zero matrix*.

zero factor property Also called zero product property. If for two numbers (real or complex) a and b, $a \cdot b = 0$, then either $a = 0$ or $b = 0$ or both are zero. This property of numbers is one of the most important tools in solving *algebraic* or *trigonometric* equations. Examples: (1) $x^3 + 3x^2 - 4x - 12 = 0$. Factoring the polynomial on the left side by grouping, we get the equation $(x-2)(x+2)(x+3) = 0$ which by the zero factor property results in three linear equations $x-2=0, x+2=0, x+3=0$ with the solutions $x = 2, x = -2, x = -3$. (2) $\sin 2x - \cos x = 0$. Using the *double angle* formula for the sine function and factoring we get $\cos x(2\sin x - 1) = 0$ and again, by the zero factor property this results in two simpler equations $\cos x = 0$ and $\sin x = 1/2$. The solutions on the base interval $[0, 2\pi)$ are: $x = \pi/2, 3\pi/2, \pi/6, 5\pi/6$.
The zero factor property is not true for other mathematical object, for example for matrices: If the product $A \cdot B$ of two matrices is the zero matrix then none of the matrices A or B are necessarily the zero matrices themselves.

zero matrix A matrix of arbitrary size $m \times n$ such that all of the entries are zeros. Example:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is the *additive identity* in the set of all matrices of the same size: Adding this matrix to any other matrix does not change it.

zeros of the equation Also called roots or solutions of the equation. Any real (or complex) number that substituted into the equation results in numeric identity. The number $x = 2$ is the zero of the equation $x^2 - x - 2 = 0$ because substituting that value into the left side of the equation results in numeric identity $0 = 0$. Every *polynomial equation* of *degree* $n \geq 1$ has exactly n zeros, if we accept complex zeros and count them according to *multiplicity*. See also Fundamental Theorem of Algebra.

zero subspace The *subspace* of a *vector space* that consists of a single *zero vector* $\mathbf{x} = \mathbf{0}$. Zero subspace is the part of any vector space.

zero transformation The (linear) transformation between any *vector spaces* V and W such that $T(\mathbf{u}) = \mathbf{0}$ for any vector $\mathbf{u} \in V$.

zero vector The vector that has all zero *components*: $\mathbf{0} = (0, 0, \cdots, 0)$. This vector plays the role of the *additive unity* for any vector space: $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$ for any vector $\mathbf{x}$ of the vector space.